Superstability of systems

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1. Introduction

Stability of linear time invariant systems is a standard topic in system theory already appearing in the pioneering work on the concept of state space of Lotfi Zadeh (see [1]). Stability here is characterized by the asymptotic exponential decay in the norm of the state or equivalently, the energy: the stability index must be negative

\[ \sigma = \lim_{t \to \infty} \frac{\log \| x(t) \|}{t} < 0, \]

where \( x(t) \) is the state at time \( t \) and \( \| \cdot \| \) denotes the norm.

It is natural to call a system “superstable” if the stability index

\[ \sigma = -\infty. \]

The question of interest to us then is: Do such systems exist (in nature) and how do we characterize them? This paper offers some partial answers to this problem.

2. Review of finite dimensional theory

The notion of stability for a linear time-invariant system with a finite dimensional state space description (going back in fact to Liapunov) is well-understood. Thus, letting \( x(t) \) denoting the state at time \( t \) in “free” response, the system is stable (or asymptotically stable) if

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\[
\lim_{t \to \infty} \log \frac{\|x(t)\|}{t} = -|\sigma| < 0. \tag{2.1}
\]

Since this would imply that

\[
\|x(t)\| \leq M e^{-(\sigma + \varepsilon)t}
\]

for every \(\varepsilon > 0\), we call this “exponential stability”.

In many cases, the norm \(\| \cdot \|^2\) can be taken to be proportional to the total energy in the system. Thus in an oscillatory system where the free response can be characterized as

\[
M \ddot{x}(t) + D \dot{x}(t) + Kx(t) = 0,
\]

where \(M > 0\), \(D \geq 0\) and \(K \geq 0\), the state is

\[
X(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}
\]

and defining the norm by

\[
\|X(t)\|^2 = [Kx(t), x(t)] + [M\dot{x}(t), \dot{x}(t)].
\]

We have that

\[
\|X(t)\|^2 = 2E(t),
\]

where \(E(t)\) is the total system energy. In this case

\[
\|x(t)\| \leq \|x(0)\| e^{-\mu t},
\]

where

\[
\mu = \frac{1}{2} \text{ smallest eigenvalue of } M^{-1}D.
\]

More generally we have asymptotic exponential decay as in (2.2).

3. Superstability

With \(x(t)\) denoting the state at time \(t \geq 0\), of the free response of a linear time invariant system we say that the system is superstable if

\[
\lim_{t \to \infty} \frac{\|x(t)\|}{t} = -\infty
\]

or

\[
\lim_{t \to \infty} \frac{\|x(t)\|}{t} < -\sigma \quad \text{for every } \sigma > 0.
\]
This would imply that
\[ \|x(t)\| \leq M e^{-\sigma t} \quad \text{for every } \sigma > 0, \ t \geq 0. \]

This notion was introduced in [2]. Our first observation is that a system with a finite dimensional state space description cannot be superstable. Indeed if the system is superstable, the Laplace transform
\[ \hat{x}(\lambda) = \int_0^\infty e^{-\lambda t} x(t) \, dt \quad (3.1) \]
defines an entire function of the complex variable \( \lambda \). But from the free response dynamics:
\[ \dot{x}(t) = Ax(t) \]
where \( A \) is a square matrix, and
\[ \hat{x}(\lambda) = (\lambda I - A)^{-1} x(0) \]
has to be a polynomial with no poles, but by (3.1)
\[ |\hat{x}(\lambda)| \to 0 \quad \text{as } \Re \lambda \to \infty \]
and we reach a contradiction.

Hence superstability can occur only in systems where the state space is not finite dimensional—in other words, \( A \) must be the infinitesimal generator of a semigroup of operators over some Banach space.

Let us also note here that recently Udwadia [3] has come up with a definition of a complementary concept—of super-instability.

4. Extinction in finite time: example

Before plunging into semigroup theory, let us look at the simplest possible type of superstability—viz., extinction in finite time:
\[ x(t) = 0 \quad \text{for } t \geq T > 0, \]
whatever the initial state \( x(0) \). (The extinction time does not depend on \( x(0) \).) It is remarkable that we can find a simple (almost canonical, as we shall discover eventually!) example—that we may call “physical” in that it is a boundary-value problem for a partial differential equation (see [4] for more on the genesis of this example).

Example
\[ \frac{\partial^2 \theta(t,s)}{\partial s^2} = c^2 \frac{\partial^2 \theta(t,s)}{\partial t^2}, \quad 0 < t, \ 0 < s < L < \infty, \]
which we shall henceforth shorten to
\[
\theta''(t,s) = c^2 \theta(t,s), \quad 0 < t, \ 0 < s < L < \infty
\]
and on the boundary:
\[
\theta(t,0) = 0, \quad c\theta'(t,L) + \dot{\theta}(t,L) = 0.
\]

For a state space description of this system, see [4]. Here we shall skip the abstract development and note that we can actually develop an explicit solution for given initial data \(\theta(0,\cdot)\) and \(\dot{\theta}(0,\cdot)\):
\[
\theta(t,s) = \frac{1}{2} \left[ \theta(0,s+ct) - \theta(0,s-ct) + \theta(0,ct-s) \right] + \frac{1}{2} \int_0^t \left[ \dot{\theta}(0,s+ct) - \dot{\theta}(0,s-ct) + \dot{\theta}(0,ct-s) \right] d\tau, \quad 0 < t, \ 0 < s < L
\]
and we can deduce readily that
\[
\theta(t,s) \equiv 0 \quad \text{for} \ t > \frac{2L}{c}, \ 0 < s < L.
\]

Udwadia in [3] provides an elegant, physically based technique of solution. We have thus extinction in finite time.

This particular example also occurs in much more general form in scattering theory [6] where it is called a “disappearing solution”—the “black box is undetected by the scattering process” or “no initial state in the box escapes to infinity.” Incidentally, it is curious that Phillips (see [6]), one of the leaders in scattering theory, using control terminology, refers to this as “uncontrollable”—we believe it should rather be “unobservable.”

It turns out that one can state a complete characterization of this type of superstability in the language of semigroup theory, drawing on a classical result of Paley–Wiener.

**Theorem 1** (Krein and Gohberg [5, p. 362]). Let \(A\) be the infinitesimal generator of a strongly continuous semigroup \(S(t), t \geq 0\). Let \(R(\lambda,A)\) denote the resolvent. Then
\[
S(t) = 0 \quad \text{for} \ t \geq T, \ T < \infty
\]
if and only if \(R(\lambda,A)\) is an entire function of order 1 and type \(T\).

**Proof.** See [5, p. 362]. □

**Remark.** We may verify this for our example as in [4].
More general examples. We note that our example is actually a primitive version of a Timoshenko model for torsional oscillation. It can be generalized to multi-dimensional Timoshenko models as in [7]. Thus we have:

\[
\begin{align*}
\ddot{v}(t,s) - c_1^2(v''(t,s)\phi'(t,s)) &= 0, \\
\ddot{\phi}(t,s) - c_2^2\phi''(t,s) - mc_1^2(v'(t,s) - \phi(t,s)) &= 0
\end{align*}
\]

Boundary conditions:

\[
\begin{align*}
c_1(v'(t,L) - \phi(t,L)) + \dot{v}(t,L) &= 0, \\
c_2\phi'(t,L) + \dot{\phi}(t,L) &= 0.
\end{align*}
\]

Initial conditions:

\[
\text{col}[[v(0,s)\phi(0,x)\dot{v}(0,s)\dot{\phi}(0,s)]], \quad 0 < s < L.
\]

Energy \(\sim\)

\[
E(t) = \frac{1}{2} \left[ c_1^2 \int_0^L \phi'(t,s)^2 \, ds + mc_1^2 \int_0^L |v'(t,s) - \phi(t,s)|^2 \, ds + \int_0^L \dot{\phi}(t,s)^2 \, ds + m \int_0^L \dot{v}(t,s)^2 \, ds \right],
\]

\[
X(t) = \text{col}[[v(t,\cdot)\phi(t,\cdot)\dot{v}(t,\cdot)\dot{\phi}(t,\cdot)]], \quad ||X(t)||^2 = 2E(t).
\]

Claim: superstable

\[
X(t) = 0, \quad t > T, \quad T = \frac{2L}{c} \quad \text{if } c_1 = c_2 = c.
\]

**Proof.** Calculate the resolvent and apply Theorem 1. \(\square\)

5. Open questions

As we can see, superstability in terms of semigroups simply means that the semigroup is quasi-nilpotent. There are many characterizations (\(\equiv\) necessary and sufficient conditions) of quasi-nilpotent semigroups. See [2]. However none of them would appear to be constructive—the Krein theorem (Theorem 1) is the closest “operational” version we have, but of course this is the special case where the semigroup is nilpotent. This raises a currently open question: Are there “physical” examples of superstability which are not of the type of extinction-in-finite-time? For some recent results, including examples, even if “artificial”, see Lumer [8,9]. See also the papers of Udwadia [3].
References