Solutions of Homework #7

8.1a. Find $\mathbf{H}$ in cartesian components at $P(2, 3, 4)$ if there is a current filament on the $z$ axis carrying 8 mA in the $a_z$ direction:

Applying the Biot-Savart Law, we obtain

$$
H_z = \frac{\int_{-\infty}^{\infty} d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{\int_{-\infty}^{\infty} dz a_z \times [2a_x + 3a_y + (4 - z)a_z]}{4\pi (z^2 - 8z + 29)^{3/2}} = \frac{\int_{-\infty}^{\infty} dz [2a_y - 3a_x]}{4\pi (z^2 - 8z + 29)^{3/2}}
$$

Using integral tables, this evaluates as

$$
H_z = \frac{I}{4\pi} \left[ \frac{2(2z - 8)(2a_y - 3a_x)}{52(z^2 - 8z + 29)^{1/2}} \right]_{-\infty}^{\infty} = \frac{I}{26\pi} (2a_y - 3a_x)
$$

Then with $I = 8$ mA, we finally obtain $H_z = -294a_x + 106a_y \mu A/m$

b. Repeat if the filament is located at $x = -1$, $y = 2$: In this case the Biot-Savart integral becomes

$$
H_y = \int_{-\infty}^{\infty} d\mathbf{l} \times \mathbf{a}_R \times [(2 + 1)a_x + (3 - 2)a_y + (4 - z)a_z] = \int_{-\infty}^{\infty} dz [3a_y - a_x]
$$

Evaluating as before, we obtain with $I = 8$ mA:

$$
H_y = \frac{I}{4\pi} \left[ \frac{2(2z - 8)(3a_y - a_x)}{40(z^2 - 8z + 26)^{1/2}} \right]_{-\infty}^{\infty} = \frac{I}{20\pi} (3a_y - a_x) = -127a_x + 382a_y \mu A/m
$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts a and b, or

$$
\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = -421a_x + 578a_y \mu A/m
$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

8.2. A filamentary conductor is formed into an equilateral triangle with sides of length $\ell$ carrying current $I$. Find the magnetic field intensity at the center of the triangle.

I will work this one from scratch, using the Biot-Savart law. Consider one side of the triangle, oriented along the $z$ axis, with its end points at $z = \pm \ell/2$. Then consider a point, $x_0$, on the $x$ axis, which would correspond to the center of the triangle, and at which we want to find $\mathbf{H}$ associated with the wire segment. We thus have $d\mathbf{l} = dz \mathbf{a}_z$, $R = \sqrt{z^2 + x_0^2}$, and $a_R = [x_0 a_y - z a_x]/R$. The differential magnetic field at $x_0$ is now

$$
d\mathbf{H} = \frac{I dz \mathbf{a}_z \times \mathbf{a}_R}{4\pi R^2} = \frac{I dz a_z \times (x_0 a_y - z a_x)}{4\pi (x_0^2 + z^2)^{3/2}} = \frac{I dz x_0 a_y}{4\pi (x_0^2 + z^2)^{3/2}}
$$

where $a_y$ would be normal to the plane of the triangle. The magnetic field at $x_0$ is then

$$
\mathbf{H} = \int_{-\ell/2}^{\ell/2} \frac{I dz x_0 a_y}{4\pi (x_0^2 + z^2)^{3/2}} = \frac{I x_0 a_y \sqrt{\ell^2 + 4x_0^2}}{4\pi x_0 \sqrt{\ell^2 + 4x_0^2}^{1/2}} = \frac{I}{2\pi} \frac{x_0 a_y}{\sqrt{\ell^2 + 4x_0^2}}
$$

8.2. (continued). Now, $x_0$ lies at the center of the equilateral triangle, and from the geometry of the triangle, we find that $x_0 = (\ell/2)\tan(30^\circ) = \ell/(2\sqrt{3})$. Substituting this result into the just-found expression for $\mathbf{H}$ leads to $\mathbf{H} = 3I/(2\pi\ell) a_y$. The contributions from the other two sides of the triangle effectively multiply the above result by three. The final answer is therefore $H_{net} = 9I/(2\pi\ell) a_y \mu A/m$. It is also possible to work this problem (somewhat more easily) by using Eq. (9), applied to the triangle geometry.
8.7. Given points $C(5,-2,3)$ and $P(4,-1,2)$; a current element $I dL = 10^{-4}(4,-3,1)$ $A \cdot m$ at $C$ produces a field $dH$ at $P$.

a) Specify the direction of $dH$ by a unit vector $a_H$: Using the Biot-Savart law, we find

$$dH = \frac{I dL \times a_{CP}}{4\pi R_{CP}^2} = \frac{10^{-4}[4a_x - 3a_y + a_z] \times [-a_x + a_y - a_z]}{4\pi 3^{3/2}} = \frac{[2a_x + 3a_y + a_z] \times 10^{-4}}{65.3}$$

from which

$$a_H = \frac{2a_x + 3a_y + a_z}{\sqrt{14}} = 0.53a_x + 0.80a_y + 0.27a_z$$

b) Find $|dH|$.

$$|dH| = \frac{\sqrt{14 \times 10^{-4}}}{65.3} = 5.73 \times 10^{-6} \text{ A/m} = 5.73 \mu\text{A/m}$$

c) What direction $a_H$ should $I dL$ have at $C$ so that $dH = 0$? $I dL$ should be collinear with $a_{CP}$, thus rendering the cross product in the Biot-Savart law equal to zero. Thus the answer is $a_H = \pm(-a_x + a_y - a_z)/\sqrt{3}$

8.10. A hollow spherical conducting shell of radius $a$ has filamentary connections made at the top $(r = a, \theta = 0)$ and bottom $(r = a, \theta = \pi)$. A direct current $I$ flows down the upper filament, down the spherical surface, and out the lower filament. Find $H$ in spherical coordinates (a) inside and (b) outside the sphere.

Applying Ampere’s circuit law, we use a circular contour, centered on the $z$ axis, and find that within the sphere, no current is enclosed, and so $H = 0$ when $r < a$. The same contour drawn outside the sphere at any $z$ position will always enclose $I$ amps, flowing in the negative $z$ direction, and so

$$H = \frac{I}{2\pi \rho} a_\phi = -\frac{I}{2\pi r \sin \theta} a_\phi \text{ A/m} \quad (r > a)$$

8.15. Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ $\text{kS/m}$. An electric field of $30 \text{a}_x$ $\text{V/m}$ is present.

a) Find $J$: Use

$$J = \sigma E = 45e^{-150\rho} a_x \text{ kA/m}$$

b) Find the total current crossing the surface $\rho = \rho_0$, $z = 0$, all $\phi$:

$$I = \int \int J \cdot dS = \int_0^{2\pi} \int_0^{\rho_0} 45e^{-150\rho} \rho d\rho d\phi = \frac{2\pi(45)}{(150)^2} e^{-150\rho_0} [-150\rho_0 - 1]_0^{\rho_0} \text{ kA}$$

$$= 12.6 [1 - (1 + 150\rho_0)e^{-150\rho_0}] \text{ A}$$

c) Make use of Ampere’s circuit law to find $H$: Symmetry suggests that $H$ will be $\phi$-directed only, and so we consider a circular path of integration, centered on and perpendicular to the $z$ axis. Ampere’s law becomes: $2\pi \rho H_\phi = I_{circ}$, where $I_{circ}$ is the current found in part b, except with $\rho_0$ replaced by the variable, $\rho$. We obtain

$$H_\phi = \frac{2.00}{\rho} [1 - (1 + 150\rho)e^{-150\rho}] \text{ A/m}$$
8.16. A balanced coaxial cable contains three coaxial conductors of negligible resistance. Assume a solid inner conductor of radius \(a\), an intermediate conductor of inner radius \(b_i\), outer radius \(b_o\), and an outer conductor having inner and outer radii \(c_i\) and \(c_o\), respectively. The intermediate conductor carries current \(I\) in the positive \(a_o\) direction and is at potential \(V_o\). The inner and outer conductors are both at zero potential, and carry currents \(I/2\) (in each) in the negative \(a_o\) direction. Assuming that the current distribution in each conductor is uniform, find:

a) \(J\) in each conductor: These expressions will be the current in each conductor divided by the appropriate cross-sectional area. The results are:

\[
\text{Inner conductor: } J_a = -\frac{I}{2\pi a^2} \ A/m^2 \quad (0 < \rho < a)
\]

\[
\text{Center conductor: } J_b = \frac{I}{\pi(b_o^2 - b_i^2)} \ A/m^2 \quad (b_i < \rho < b_o)
\]

\[
\text{Outer conductor: } J_c = -\frac{I}{2\pi(c_o^2 - c_i^2)} \ A/m^2 \quad (c_i < \rho < c_o)
\]

8.16b) \(H\) everywhere:

For \(0 < \rho < a\), and with current in the negative \(z\) direction, Ampere’s circuit law applied to a circular path of radius \(\rho\) within the given region leads to

\[
2\pi \rho H = -\frac{\rho^2}{\pi} J_a = -\frac{\rho^2 I}{2\pi a^2} \quad \Rightarrow \quad H_1 = -\frac{I}{4\pi a^2} a_o \ A/m \quad (0 < \rho < a)
\]

For \(a < \rho < b_i\), and with the current within in the negative \(z\) direction, Ampere’s circuital law applied to a circular path of radius \(\rho\) within the given region leads to

\[
2\pi \rho H = -I/2 \Rightarrow H_2 = -\frac{I}{4\pi \rho} a_o \ A/m \quad (a < \rho < b_i)
\]

Inside the center conductor, the net magnetic field will include the contribution from the inner conductor current:

\[
2\pi \rho H = -I/2 + \frac{\pi \rho^2}{\pi(b_o^2 - b_i^2)} J_b = -\frac{I}{4\pi \rho} \left[ \frac{2(\rho^2 - b_i^2)}{(b_o^2 - b_i^2)} - 1 \right] a_o \ A/m \quad (b_i < \rho < b_o)
\]

Beyond the center conductor, but before the outer conductor, the net enclosed current is \(I - I/2 = I/2\), and the magnetic field is

\[
H_3 = -\frac{I}{4\pi \rho} a_o \quad (b_o < \rho < c_i)
\]

Inside the outer conductor (with current again in the negative \(z\) direction) the field associated with the outer conductor current will subtract from \(H_4\) (more so as \(\rho\) increases):

\[
H_4 = -\frac{I}{4\pi \rho} \left[ 1 - \frac{\rho^2}{(c_o^2 - c_i^2)} \right] a_o \ A/m \quad (c_i < \rho < c_o)
\]

Finally, beyond the outer conductor, the total enclosed current is zero, and so

\[
H_6 = 0 \quad (\rho > c_o)
\]
c) Electric everywhere: Since we have perfect conductors, the electric field within each will be zero. This leaves the free space regions, within which Laplace’s equation will have the general solution form, \( V(\rho) = C_1 \ln(\rho/a) + C_2 \). Between radii \( a \) and \( b \), the boundary condition, \( V = 0 \) at \( \rho = a \) leads to \( C_2 = -C_1 \ln a \). Thus \( V(\rho) = C_1 \ln(\rho/a) \). The boundary condition, \( V = V_0 \) at \( \rho = b \), leads to \( C_1 = \frac{V_0}{\ln(b/a)} \), and so finally, \( V(\rho) = \frac{V_0}{\ln(\rho/a)} / \ln(b/a) \). Now

\[
E_1 = -\nabla V = -\frac{dV}{d\rho} a_\rho = -\frac{V_0}{\rho \ln(b/a)} a_\rho \ V / m \quad (a < \rho < b)
\]

Between radii \( b \) and \( c \), the boundary condition, \( V = 0 \) at \( \rho = c \), leads to \( C_2 = -C_1 \ln c \). Thus \( V(\rho) = c_1 \ln(\rho/c) \). The boundary condition, \( V = V_0 \) at \( \rho = b \), leads to \( C_1 = \frac{V_0}{\ln(b/c)} \), and so finally, \( V(\rho) = \frac{V_0}{\ln(\rho/c)} / \ln(b/c) \). Now

\[
E_2 = -\frac{dV}{d\rho} a_\rho = -\frac{V_0}{\rho \ln(b/c)} a_\rho = \frac{V_0}{\rho \ln(c/b)} \ a_\rho \ V / m \quad (b < \rho < c)
\]

8.19. Calculate \( \nabla \times [\nabla (\nabla \cdot \mathbf{G})] \) if \( \mathbf{G} = 2xz \mathbf{a}_x - 20y \mathbf{a}_y + (x^2 - z^2) \mathbf{a}_z \): Proceeding, we first find \( \nabla \cdot \mathbf{G} = 4xy z - 20 - 2z \). Then \( \nabla (\nabla \cdot \mathbf{G}) = 4yz \mathbf{a}_x + 4xz \mathbf{a}_y + (4xy - 2) \mathbf{a}_z \). Then

\[
\nabla \times [\nabla (\nabla \cdot \mathbf{G})] = (4x - 4x) \mathbf{a}_x - (4y - 4y) \mathbf{a}_y + (4z - 4z) \mathbf{a}_z = 0
\]

8.23. Given the field \( \mathbf{H} = 20 \rho^2 \mathbf{a}_\phi \ A/m \):

a) Determine the current density \( \mathbf{J} \): This is found through the curl of \( \mathbf{H} \), which simplifies to a single term, since \( \mathbf{H} \) varies only with \( \rho \) and has only a \( \phi \) component:

\[
\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho \mathbf{a}_\phi) = \frac{1}{\rho} \frac{d}{d\rho} (20 \rho^3) \mathbf{a}_z = 60 \rho \mathbf{a}_z \ A/m^2
\]

b) Integrate \( \mathbf{J} \) over the circular surface \( \rho = 1, \ 0 < \phi < 2\pi, \ z = 0 \), to determine the total current passing through that surface in the \( \mathbf{a}_z \) direction: The integral is:

\[
I = \int_0^{2\pi} \left[ \int_0^1 60 \rho \mathbf{a}_z \cdot \rho d\rho \right] d\phi = 40\pi \ A
\]

c) Find the total current once more, this time by a line integral around the circular path \( \rho = 1, \ 0 < \phi < 2\pi, \ z = 0 \):

\[
I = \oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} 20 \rho^2 \mathbf{a}_\phi \ |_{\rho=1} \cdot (1)d\phi = 40\pi \ A
\]
8.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find $\mathbf{J}$ within the conductor: Assuming the current is $+z$ directed, 
\[
\mathbf{J} = \frac{2}{\pi (0.2 \times 10^{-3})^2} a_z = 1.59 \times 10^7 \text{ A/m}^2
\]

b) Use Ampere’s circuital law to find $\mathbf{H}$ and $\mathbf{B}$ within the conductor: Inside, at radius $\rho$, we have 
\[
2\pi \rho H_\rho = \pi \rho^2 J \quad \Rightarrow \quad H = \frac{\rho J}{2} a_\phi = 7.96 \times 10^6 \rho a_\phi \text{ A/m}
\]

Then $\mathbf{B} = \mu_0 \mathbf{H} = (4\pi \times 10^{-7})(7.96 \times 10^6) a_\phi = 10\rho a_\phi \text{ Wb/m}^2$.

c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part b, we find, 
\[
\nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi) a_z = \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{1.59 \times 10^7 \rho^2}{2} \right) a_z = 1.59 \times 10^7 a_z \text{ A/m}^2 = \mathbf{J}
\]

d) Find $\mathbf{H}$ and $\mathbf{B}$ outside the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius $\rho$, and so 
\[
\mathbf{H} = \frac{I}{2\pi \rho} a_\phi = \frac{1}{\pi \rho} a_\phi \text{ A/m}
\]

Now $\mathbf{B} = \mu_0 \mathbf{H} = \mu_0/\pi \rho a_\phi \text{ Wb/m}^2$.

e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor: Here we use $\mathbf{H}$ outside the conductor and write: 
\[
\nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi) a_z = \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{1}{\pi \rho} \right) a_z = 0 \quad (\text{as expected})
\]