3.5. Let \( \mathbf{D} = 4\gamma \mathbf{a}_x + 2(\mathbf{r}^2 + z^2)\mathbf{a}_y + 4y\mathbf{a}_z \) C/m\(^2\) and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped \( 0 < x < 2, 0 < y < 3, 0 < z < 5 \) m. Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at \( y = 0 \) and \( z = 3 \). The \( y \) component of \( \mathbf{D} \) will penetrate those surfaces, but will be inward at \( y = 0 \) and outward at \( y = 3 \), while having the same magnitude in both cases. These fluxes will thus cancel. At the \( x = 0 \) plane, \( D_x = 0 \) and at the \( z = 0 \) plane, \( D_z = 0 \), so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at \( x = 2 \) and \( z = 5 \). The net outward flux becomes:

\[
\Phi = \int_0^3 \int_0^3 \mathbf{D}_{\mathbf{r}} \cdot \mathbf{a}_x \, dy \, dz + \int_0^3 \int_0^2 \mathbf{D}_{\mathbf{r}} \cdot \mathbf{a}_x \, dx \, dy
\]

\[
= 5 \int_0^3 4(2)yd\,dy + 2 \int_0^3 4(5)y\,dy = 360 \text{ C}
\]

3.7. Volume charge density is located in free space as \( \rho_v = 2e^{-1000r} \) nC/m\(^3\) for \( 0 < r < 1 \) mm, and \( \rho_v = 0 \) elsewhere.

\( a \) Find the total charge enclosed by the spherical surface \( r = 1 \) mm: To find the charge we integrate:

\[
Q = \int_0^{2\pi} \int_0^\pi \int_0^{0.001} 2e^{-1000r^2} \sin \theta \, d\theta \, dr \, d\phi
\]

Integration over the angles gives a factor of \( 4\pi \). The radial integration we evaluate using tables; we obtain

\[
Q = 8\pi \left[ \frac{-r^2e^{-1000r}}{1000} \bigg|_0^{0.001} + \frac{2}{1000(1000)^2} \left( -1000r - 1 \right) \bigg|_0^{0.001} \right] = 4.0 \times 10^{-9} \text{ nC}
\]

\( b \) By using Gauss’s law, calculate the value of \( D_r \) on the surface \( r = 1 \) mm: The gaussian surface is a spherical shell of radius \( 1 \) mm. The enclosed charge is the result of part \( a \). We thus write \( 4\pi r^2 D_r = Q \), or

\[
D_r = \frac{Q}{4\pi r^2} = \frac{4.0 \times 10^{-9}}{4\pi(0.001)^2} = 3.2 \times 10^{-4} \text{ nC/m}^2
\]

Remarks: Get familiar with these commonly used integrals!

3.12. The sun radiates a total power of about \( 2 \times 10^{26} \) watts (W). If we imagine the sun’s surface to be marked off in latitude and longitude and assume uniform radiation, (a) what power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W? (b) What is the power density on a spherical surface 93,000,000 km from the sun in W/m\(^2\)?

\( a \) The power density on a spherical surface with radius \( r \) is

\[
p_r = \frac{W}{4\pi r^2} = \frac{2 \times 10^{26}}{4\pi r^2} \text{ W/m}^2\]

\[
P = \int_{-\frac{27}{180}}^{\frac{27}{180}} \int_{-\frac{60}{180}}^{\frac{60}{180}} \int_0^{2\pi} \int_0^{0.001} p_r^2 \sin \theta d\theta d\phi d\theta = \frac{2 \times 10^{26}}{4\pi} \times \frac{(27-12)\times \pi}{180} \times (-\cos \theta) \bigg|_{0}^{\pi} = 5.95 \times 10^{23} \text{ W}
\]

\( b \) The power density is

\[
p_r = \frac{W}{4\pi r^2} = \frac{2 \times 10^{26}}{4\pi(93 \times 10^6 \times 1609)} = 710.8 \text{ W/m}^2
\]
3.15. Volume charge density is located as follows: \( \rho_v = 0 \) for \( \rho < 1 \) mm and for \( \rho > 2 \) mm, \( \rho_v = 4 \rho \mu \text{C/m}^2 \) for \( 1 < \rho < 2 \) mm.

  a) Calculate the total charge in the region \( 0 < \rho < \rho_1, 0 < z < L \), where \( 1 < \rho_1 < 2 \) mm:

  We find
  \[
  Q = \int_0^L \int_0^{2\pi} \int_0^{\rho_1} 4 \rho \rho \, dho \, d\phi \, dz = \frac{8\pi L}{3} [\rho_1^3 - 10^{-9}] \mu \text{C}
  \]

  where \( \rho_1 \) is in meters.

  b) Use Gauss’ law to determine \( D_\rho \) at \( \rho = \rho_1 \): Gauss’ law states that \( 2\pi \rho_1 LD_\rho = Q \), where \( Q \) is the result of part a. Thus
  \[
  D_\rho(\rho_1) = \frac{4(\rho_1^3 - 10^{-9})}{3\rho_1} \mu \text{C/m}^2
  \]

  where \( \rho_1 \) is in meters.

  c) Evaluate \( D_\rho \) at \( \rho = 0.8 \) mm, 1.6 mm, and 2.4 mm: At \( \rho = 0.8 \) mm, no charge is enclosed by a cylindrical gaussian surface of that radius, so \( D_\rho(0.8\text{mm}) = 0 \). At \( \rho = 1.6 \) mm, we evaluate the part b result at \( \rho_1 = 1.6 \) to obtain:

  \[
  D_\rho(1.6\text{mm}) = \frac{4((.0016)^3 - (.0010)^3)}{3(.0016)} = 3.6 \times 10^{-6} \mu \text{C/m}^2
  \]

  At \( \rho = 2.4 \), we evaluate the charge integral of part a from .001 to .002, and Gauss’ law is written as

  \[
  2\pi \rho L D_\rho = \frac{8\pi L}{3} [(0.002)^2 - (0.001)^2] \mu \text{C}
  \]

  from which \( D_\rho(2.4\text{mm}) = 3.9 \times 10^{-6} \mu \text{C/m}^2 \).

3.16. In spherical coordinates, a volume charge density \( \rho_v = 10 e^{-2r} \) C/m\(^3\) is present. (a) Determine \( \mathbf{D} \). (b) Check your result of part a by evaluating \( \nabla \cdot \mathbf{D} \).

  a) Use Gauss’ law, and we know only \( D_r \) component exists

  \[
  D_r \times 4\pi r^2 = \int_0^2 \int_0^\pi \int_0^2 [10 e^{-2r}] r^2 \sin \theta \, d\rho \, d\theta \, d\phi
  \]

  \[
  \bar{D} = \left( -\frac{r^2}{2} e^{-2r} - \frac{r}{2} e^{-2r} - \frac{1}{4} e^{-2r} + \frac{1}{4} \right) \hat{r}
  \]

  b) Using the formula of divergence in the spherical coordinate system, we will have

  \[
  \nabla \cdot \bar{D} = 10 e^{-2r} = \rho_v
  \]

  The integral in radial direction is very similar to prob. 3.7(a). Here we use the integral form of Gauss’ law to easily obtain the electric flux density \( \mathbf{D} \), and the result is double checked by the differential form of Gauss’ law.

3.19. A spherical surface of radius 3 mm is centered at \( P(4, 1, 5) \) in free space. Let \( \mathbf{D} = x \hat{a}_x \) C/m\(^2\). Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface: We use \( \Phi = \nabla \cdot \mathbf{D} \Delta \mathbf{v} \), where in this case \( \nabla \cdot \mathbf{D} = (\partial/\partial x) x = 1 \) C/m\(^3\). Thus

  \[
  \Phi = \frac{4}{3} \pi (3)^2 (1) = 1.13 \times 10^{-7} \text{C} = 113\text{nC}
  \]

  Remarks: A direct surface integral of electric flux density is not an easy way!
3.21. Calculate the divergence of \( \mathbf{D} \) at the point specified if

a) \( \mathbf{D} = (1/z^2) \left[ 10xyz \mathbf{a}_x + 5x^2z \mathbf{a}_y + (2z^3 - 5x^2y) \mathbf{a}_z \right] \) at \( P(-2, 3, 5) \): We find

\[
\nabla \cdot \mathbf{D} = \left[ \frac{10y}{z} + 0 + 2 + \frac{10x^2y}{z^3} \right]_{(-2, 3, 5)} = 8.96
\]

b) \( \mathbf{D} = 5x^2 \mathbf{a}_x + 10yz \mathbf{a}_z \) at \( P(3, -45^\circ, 5) \): In cylindrical coordinates, we have

\[
\nabla \cdot \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \rho_{\rho}) + \frac{1}{\rho} \frac{\partial \rho_{\phi}}{\partial \phi} + \frac{\partial \rho_z}{\partial z} = \left[ \frac{5x^2}{\rho} + 10 \rho \right]_{(3, -45^\circ, 5)} = 71.67
\]

c) \( \mathbf{D} = 2r \sin \theta \sin \phi \mathbf{a}_r + r \cos \theta \sin \phi \mathbf{a}_\theta + r \cos \phi \mathbf{a}_\phi \) at \( P(3, 45^\circ, -45^\circ) \): In spherical coordinates, we have

\[
\nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta D_\theta) + \frac{1}{r \sin \phi} \frac{\partial D_\phi}{\partial \phi} = \left[ 6 \sin \theta \sin \phi + \frac{\cos 2\theta + \sin \phi}{\sin \theta} - \frac{\sin \phi}{\sin \theta} \right]_{(3, 45^\circ, -45^\circ)} = -2
\]

3.25. Within the spherical shell, \( 3 < r < 4 \) m, the electric flux density is given as

\[
\mathbf{D} = 5(r - 3)^3 \mathbf{a}_r \text{ C/m}^2
\]

a) What is the volume charge density at \( r = 4 \)? In this case we have

\[
\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{d}{dr} (r^2 D_r) = \frac{5}{r} (r - 3)^2 (5r - 6) \text{ C/m}^3
\]

which we evaluate at \( r = 4 \) to find \( \rho_v(r = 4) = 17.50 \text{ C/m}^3 \).

b) What is the electric flux density at \( r = 4 \)? Substitute \( r = 4 \) into the given expression to find \( \mathbf{D}(4) = 5 \mathbf{a}_r \text{ C/m}^2 \).

c) How much electric flux leaves the sphere \( r = 4 \)? Using the result of part b, this will be

\[
\Phi = 4\pi (4)^2 (5) = 320 \pi \text{ C}
\]

d) How much charge is contained within the sphere, \( r = 4 \)? From Gauss’ law, this will be the same as the outward flux, or again, \( Q = 320 \pi \text{ C} \).