2) The current density is given as

\[ J = \exp\left(-10^3 s^{-1} \cdot t\right) \cdot \frac{2}{r} \cdot \frac{A}{m} \]

a) How much current is crossing the surface \( r = 10 m \) at the line \( t = 1 \text{ ms} \)?

\[
I = \oint J \, dS = J \oint dS = \exp\left(-\frac{10^3 \cdot 10^3}{r_0}ight) \cdot \frac{2}{10m} \cdot 4\pi 10^2 m^2 \frac{A}{m} = 92.5 A
\]

b) Repeat for \( r = 4 m \)

\[
I = \exp(-1) \cdot \frac{2}{4m} \cdot 4\pi 4^2 m^2 \frac{A}{m} = 2.9 A
\]

c) Use the continuity equation to find \( P_v(r,t) \), under the assumption that \( P_v \to 0 \) as \( t \to \infty \)

\[
\nabla \cdot J = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{2}{r} \exp\left(-10^3 s^{-1} t\right) \frac{A}{m} \right) = \frac{2}{r^2} \exp\left(-10^3 s^{-1} t\right) \frac{A}{m} = -\frac{\partial P_v}{\partial t}
\]

\[
P_v (r,t) = -\int \frac{2}{r^2} \exp\left(-10^3 s^{-1} t\right) \frac{A}{m} \, dt + f(r)
\]

\[
= \frac{2 \cdot 10^{-3}}{r^2} \cdot \exp\left(-10^3 s^{-1} t\right) \frac{As}{m} + f(r)
\]

If \( t \to \infty \) then \( P_v \to 0 \) \( \Rightarrow f(r) = 0 \)

\[
\Rightarrow P_v (r,t) = \frac{2 \cdot 10^{-3}}{r^2} \cdot \exp\left(-10^3 s^{-1} t\right) \frac{As}{m}
\]
d) What is the charge density at the surface and line from a)

\[ r = 10 \text{ m}, \quad t = 10^{-3} \text{ s} \]

\[ \Rightarrow \rho_v = \frac{2 \times 10^{-3}}{(10 \text{ m})^2} \cdot \exp(-1) \frac{A}{m} = 7.4 \times 10^{-6} \frac{C}{m^3} = 7.4 \frac{C}{cm^3} \]

\( F \) How can it be possible that more current leaves the \( r = 10 \text{ m} \) surface, than the \( r = 4 \text{ m} \) surface?

\( V(10 \text{ m}) = 10^4 \frac{m}{s}, \quad V(4 \text{ m}) = 4 \times 10^{-3} \frac{m}{s} < V(\text{km}) \)

\( \Rightarrow \) the charges are accelerated.

This acceleration leads to \( I_2 > I_1 \), while the total charge is conserved.

\( \rho \) Find an expression for the velocity of the charge density

\[ v = \frac{J}{\rho_v} = \frac{\frac{2}{r} \cdot \exp(-10^3 s + t) \frac{A}{m} \frac{A}{s}}{2 \times 10^{-3} r^2 \cdot \exp(-10^3 s + t) \cdot \frac{A}{m}} \]

\[ = 10^3 r \cdot \frac{A}{s} \frac{1}{s} \]
two concentric cylindrical conductors in free space

\[ r_a = 0.01 \text{ m}, \quad r_b = 0.08 \text{ m} \]

have surface charge densities \( P_{s,a} = 40 \frac{\text{C}}{\text{m}^2} \) and \( P_{s,b} \) such that \( D \) and \( E \) fields exist between the two cylinders but are zero elsewhere.

Find \( D \) and \( E \) between the cylinders.

By symmetry, the field between the cylinders must be radial and a function of \( r \) only.

For \( r_c < r < r_b \)

\[ \mathbf{E} \cdot \mathbf{r} = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot D_r) = 0 \]

for every \( r \Rightarrow r \cdot D_r \) must be constant

\[ r \cdot D_r = c \]

To get \( c \)

\[ D_r \cdot 4\pi r^2 = \mathbf{A} \]

\[ D_r = \frac{\mathbf{A}}{4\pi r^2} = P_c(r) \]

at \( r = r_a + a \)

\[ D_r = 40 \cdot 10^{-12} \frac{\text{C}}{\text{m}^2} \]

\[ C = r \cdot D_r = 0.01 \text{ m} \cdot 40 \cdot 10^{-12} \frac{\text{C}}{\text{m}^2} = 4 \cdot 10^{-13} \frac{\text{C}}{\text{m}} \]

and so \( D = \frac{4 \cdot 10^{-13}}{r} \frac{\text{C}}{\text{m}} \)
\[ E = \frac{D}{\varepsilon_0} = \frac{4.52 \cdot 10^{-2}}{r} \text{ V} \]

\[ P_{sh} = -\frac{D}{\varepsilon_0} = -\frac{4.6^{-13}}{0.08} \frac{c}{m^2} = -5 \frac{pc}{m^2} \]
3) A cylindrical wire of conductivity $\sigma_1$, and radius $r_1$, is surrounded by a cylindrical jacket of conductivity $\sigma_2$ (inner radius $r_1$, outer radius $r_2$, $r_2 > r_1$).

Show that the ratio of the current densities in the two materials is independent of $r_1$ and $r_2$.

A constant voltage between the two ends of the wire means that the field within must be constant throughout the wire cross section.

\[ E = \frac{J_1}{\sigma_1} = \frac{J_1}{\sigma_2} \]

\[ \Rightarrow \frac{J_1}{J_2} = \frac{\sigma_1}{\sigma_2} \quad \text{independent of } r_1, r_2 \]

q.e.d.
A brass tube has an inner radius of 2 cm, and a wall thickness of 1 mm, and a length of 1 m. The conductivity of brass is $\sigma = 1.5 \cdot 10^{-7} \ \text{S/m}$.

A current of 150 A dc is flowing down the tube.

(a) What is the voltage drop across a 1 m length of the tube?

$$R = \frac{L}{\sigma \cdot A} = \frac{1 \text{ m}}{1.5 \cdot 10^{-7} \ \text{S/m} \cdot (\pi (2.1 \text{ cm})^2 - \pi (2 \text{ cm})^2)}$$

$$= 5.2 \cdot 10^{-4} \ \Omega$$

$$V = \frac{I}{R} = 150 \text{ A} \cdot 5.2 \cdot 10^{-4} \ \Omega = 77.6 \ \text{mV}$$

(b) What is the voltage drop if the interior of the tube is filled with a conducting material of $\sigma = 1.5 \cdot 10^5 \ \text{S/m}$.

$$R_{\text{inner}} = \frac{1 \text{ m}}{1.5 \cdot 10^5 \ \text{S/m} \cdot \pi (2 \text{ cm})^2} = 5.3 \cdot 10^{-3} \ \Omega$$

Total resistance is

$$R_T = \frac{R_1 \cdot R_2}{R_1 + R_2} = 0.00047 \ \Omega$$

$$\Rightarrow V = 150 \text{ A} \cdot 0.00047 \ \Omega = 71 \ \text{mV}$$
5.) Prove that two resistors $R_1$ and $R_2$ in parallel behave like one resistor with a total resistance of $R_{\text{tot}} = \frac{R_1 \cdot R_2}{R_1 + R_2}$.

\[ V = \frac{L}{b \cdot s} \cdot I \]

Two resistors in series see the same potential drop. The total current through both resistors is

\[ I = I_1 + I_2 \]

\[ I_1 = \frac{V \cdot b_1 \cdot S_1}{L_1} \]

\[ I_2 = \frac{V \cdot b_2 \cdot S_2}{L_2} \]

\[ I = V \left( \frac{b_1 \cdot S_1}{L_1} + \frac{b_2 \cdot S_2}{L_2} \right) = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \]

\[ \Rightarrow \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R = \frac{R_1 \cdot R_2}{R_1 + R_2} \]
6.) A battery of voltage \( V \) is attached across a cylindrical wire resistor of length \( L \) and cross-sectional area \( A \).

If I double the length of the resistor but attach it to the same battery, what happens to the drift velocity of electrons \( v_d \) in the new resistor (compared to the short resistor)?

Length increases, but potential difference stays the same.

\[ E = \frac{V}{L}, \text{ hence } E \text{ decreases} \]

\[ v_d \propto E, \text{ hence } v_d \text{ decreases} \]

6.) A battery of voltage \( V \) is attached across a cylindrical wire of length \( L \), and cross-sectional area \( A \), causing a current \( I \). If I double the area of the resistor (but attach it to the same battery), what happens to the drift velocity of electrons \( v_d \) in the new resistor?

Larger area \( \Rightarrow \) total resistance decreases, current increases.

However, \( E \) does not change (same potential over same length),

\( \Rightarrow \) drift velocity of electrons does not change.
c) How does the resistivity of a cube of conducting material (conductivity \( \sigma \)) change, if we increase its volume by a factor of \( n \)?

\[ a = \text{length of side} \Rightarrow V_1 = a^3 \]

\[ R_1 = \frac{a}{a^2 \cdot \sigma} = \frac{1}{a \sigma} = \frac{1}{\sqrt[3]{V_1}} \]

\[ a_2 = n \cdot a \]

\[ R_2 = \frac{V_2}{(n \cdot a)^2 \cdot \sigma} = \frac{1}{n^3} \]

\[ R_2 = \frac{(n \cdot a)^2 \cdot \sigma}{n^3} = \frac{1}{na^2} \]

\[ \frac{R_2}{R_1} = \frac{a^2}{na^2} = \frac{1}{n}, \quad R_2 = \frac{R_1}{n} \]

\[ \Rightarrow R_2 \text{ is smaller by a factor of } n \]
A potential field in free space is given as

$$V(x, z) = \frac{10xz}{2 + x^2}$$

in free space.

a) What is $\mathbf{D}$ at the surface $z = 0$?

$$E = -\mathbf{D} = -10z \cdot \frac{d}{dx} \left( \frac{x}{2 + x^2} \right) \mathbf{a}_x = -\frac{10x}{2 + x^2} \mathbf{a}_x \mathbf{V}_m$$

$$E(z = 0) = -\frac{10x}{2 + x^2} \mathbf{a}_z \mathbf{V}_m$$

$$\mathbf{D}(z = 0) = -\frac{10 \varepsilon_0 x}{2 + x^2} \mathbf{a}_z \frac{C}{m^2}$$

b) Show that the surface ($z = 0$) is an equipotential surface.

$$V(x, z = 0) = 0 \text{ for all } x$$

Also,

$$E(z = 0) \propto \mathbf{a}_z \text{ only } \Rightarrow \text{ always perpendicular to surface}$$

c) If the $z = 0$ surface was a conductor, what would be the total charge on the $p$ surface defined by $0 < x < 2$ and $-3 < y < 0$?

$$\mathbf{S}_S = \mathbf{D} \cdot \mathbf{a}_z \bigg|_{z = 0} = -\frac{10 \varepsilon_0 x}{2 + x^2} \frac{C}{m^2}$$
\[ Q = \int_0^2 dx \int_{y=-3}^{y=3} dy \left(- \frac{10 \varepsilon_0 x}{2 + x^2}\right) \]

\[ = -3 \cdot 10 \varepsilon_0 \left( \frac{1}{2} \right) \cdot \ln \left( \frac{x^2 + 2}{0} \right) = -15 \varepsilon_0 \left[ \ln(6) - \ln(2) \right] 
\]

\[ = -0.15 \text{ nC} \]
8.) In a cylindrical conductor of radius 2\( \text{mm} \), the current density varies with the distance \( r \) from the axis as

\[
J = 10^3 e^{-400r} \quad \text{A/m}^2
\]

Find the total current \( I \)

\[
I = \int J \, ds = \int J \, ds = \oint_{2\pi r} \, d\phi \oint_0^{0.002} \, dp \cdot 10^3 p \cdot \exp(-400p)
\]

\[
= 2\pi \cdot 10^3 \left[ \frac{\exp(-400p)}{(-400)^2} \right]_0^{0.002}
\]

\[
= 7.51 \text{ mA}
\]
Given \( J = 10^3 \text{ sin} \theta \frac{A}{m^2} \), find the current crossing the spherical shell \( r = 0.02 \text{ m} \)

\[
ds^2 = r^2 \text{ sin} \theta \ d\theta \ d\phi \ dr
\]

\( I \) and \( ds \) are both radial

\[
I = \int \int I \ ds = \int \int I \ ds = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left( 10^3 \text{ sin} \theta \right)^2 \ r^2
\]

\[
= (0.02 \text{ m})^2 \times 10^3 \times 2\pi \times \int_0^{\pi} d\theta \ (\text{ sin} \theta)^2
\]

\[
(\text{ sin} \theta)^2 = \frac{1}{2} \left( 1 - \cos(2\theta) \right)
\]

\[
= \frac{1}{2} \left[ \theta - \frac{1}{2} \sin(2\theta) \right]_0^{\pi}
\]

\[
= \frac{1}{2} \left[ \pi - 0 \right] = \frac{\pi}{2}
\]

\[
I = 3.95 \ A
\]
(b) An infinitely thin circular ring of radius 2m is charged to 10\( \mu \)C. The charge is uniformly distributed around the circular ring.

(a) Find the potential at a point on the symmetry axis at the ring, 5m from the plane of the ring.

\[
V = \int \frac{\rho \, dl}{4\pi \varepsilon_0 R}
\]

Where \( R \) is the distance from each incremental point on the ring and the point where \( V \) is to be determined.

\[
\rho \, dl = \frac{4\pi \times 2 \times 10^{-9} \, C}{2\pi \, \bar{r}} = \frac{40 \times 10^{-9} \, C}{2\pi \times (2m)} = \frac{10^{-8}}{\pi} \, \frac{C}{m}
\]

\[
R = \sqrt{r^2 + 5^2} = \sqrt{29} \, m
\]

\[
dl = r \, d\phi
\]

\[
V = \int d\phi \frac{10^{-8}}{\pi} \frac{\sqrt{29}}{m \cdot 2m} = \frac{66.9 \, V}{4\pi \varepsilon_0 \cdot \sqrt{29} \, m}
\]
b.) Compare with the result when all the charge is at the origin in favor of a point charge.

\[ V = \frac{40 \times 10^{-9} \text{ C}}{4 \pi \varepsilon_0 (5 \text{ m})^2} = 72.0 \text{ V} \]