Prob. 8.10 (Quiz #6)

(a). \( \oint C \mathbf{H} \cdot d\mathbf{l} = I \).

Inside the shell, \( \mathbf{H} = 0 \) since no current is enclosed if the closed contour is inside the shell.

(b). The geometry is symmetric in \( \phi \)-direction, so we can assume the observation point \( P \) is located at \((r, \theta, 0)\), \( r > a \).

For any differential current element located at \((a, \theta', \phi')\) in the right side of the \(xz\)-plane, \((0 < \theta' < \pi, 0 < \phi' < \pi)\). We can find the image in the left side at \((a, \theta', -\phi')\). Now we want to calculate the total \( d\mathbf{H} \) at point \( P \) due to this pair of differential current elements.

All the following calculations are performed in the rectangular coordinate system!!
Left side:

\[ \vec{R} = \begin{pmatrix} r \sin \theta \\ 0 \\ r \cos \theta \end{pmatrix} \]

\[ \vec{R}' = \begin{pmatrix} a \sin \theta \cos \phi' \\ -a \cos \theta \sin \phi' \\ a \cos \theta \end{pmatrix} \]

\[ \vec{R} = \vec{R} - \vec{R}' = \begin{pmatrix} r \sin \theta - a \sin \theta \cos \phi' \\ a \sin \theta \sin \phi' - r \cos \theta \\ a \cos \theta - a \cos \theta \end{pmatrix} \]

\[ d\vec{L} = a\theta' \vec{a}_\theta = \begin{pmatrix} \cos \theta' \cos \phi' \\ -\cos \theta' \sin \phi' \\ -\sin \theta' \end{pmatrix} d\theta' \]

\[ d\vec{L} \times \vec{R} = \begin{pmatrix} -r \cos \theta \cos \theta' \sin \phi' + a \cos \phi' \\ -r \sin \theta \sin \theta' - r \cos \theta \cos \theta' \cos \phi' + a \cos \phi' \\ r \sin \theta \cos \theta' \sin \phi' \end{pmatrix} \]

\[ R = |\vec{R}| \implies \text{equal.} \quad \iff \quad R = |\vec{R}| \]

\[ d\vec{H} = \frac{I' d\vec{L} \times \vec{R}}{4\pi R^3} \quad (I' = \frac{I}{2\pi a \sin \theta'}) \quad d\vec{H} = \frac{I' d\vec{L} \times \vec{R}}{4\pi R^3} \]

\[ d\vec{H}_{\text{total}} = \frac{I'}{4\pi R^3} \begin{pmatrix} 0 \\ -r \sin \theta \sin \phi' - r \cos \theta \cos \theta' \cos \phi' + a \cos \phi' \\ 0 \end{pmatrix} \]
Therefore, the total $d\vec{A}_{\text{total}}$ has only one component in y-direction, and it's basically $\phi$-direction considering the observation point $P$ is always in $\phi=0$ plane. The total $\vec{H}$ at $P$ is

$$\vec{H} = \int_0^\pi \int_0^{2\pi} dH_{\text{total}} \ d\theta \ d\phi,$$

and it still has $\phi$-component only.

Then we can apply Ampere's law: choose a circle as the integral path:

$$\oint \vec{H} \cdot d\ell = \int_0^{2\pi} \vec{H} \cdot \vec{\hat{\alpha}} \cdot \sqrt{R^2 + \phi^2} \ d\phi = H \int_0^{2\pi} R \sin \theta \ d\phi$$

$$= H \cdot 2\pi R \sin \theta = -I \ (\text{The current is flowing in } -\phi \text{ direction, so we have a } "-" \text{ sign here}.)$$

$$\vec{H} = H \vec{\hat{\phi}} = -\frac{I}{2\pi R \sin \theta} \vec{\hat{\phi}}$$