Electric field of a line charge

E.g., charged wire of very small radius along z-axis

Definition: \( P_L := \lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l} \) line charge density

\[ Q = \int P_L \, dl \]

To find best coordinate system ask yourself:
- In which coordinates does the field not vary?
- Which components of the field are zero?

Here, field magnitude independent of \( \phi \) and \( \rho \)

\( \Rightarrow \) Choose cylindrical coordinates

\[ dE = \frac{P_L \, dl'}{4 \pi \epsilon_0} \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

where \( \mathbf{r} = y \hat{a}_y = \rho \hat{a}_\rho \)

for this particular selection of \( P \)

\[ \mathbf{r}' = 2' \hat{a}_z \]

\[ \mathbf{r} - \mathbf{r}' = \rho \hat{a}_\rho - 2' \hat{a}_z \] and \[ |\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 - 2'^2} \]
\[ dE = \frac{P_l \, d\vec{z}'}{4\pi\varepsilon_0} \left( \frac{P_{\vec{r}_p} - \vec{z}' \cdot \vec{a}_z'}{(r^2 + \vec{z}'^2)^{3/2}} \right) \]

Only the \( E_{\vec{z}} \) component is present

\[ \Rightarrow \text{simplify to} \quad dE_p = \frac{P_l \, P \, d\vec{z}'}{4\pi\varepsilon_0 \, (r^2 + \vec{z}'^2)^{3/2}} \]

Now sum all small line charge elements:

\[ E_p = \lim_{\Delta z' \to 0} \sum_{2=-\infty}^{+\infty} dE_p = \int_{-\infty}^{\infty} dE_p \]

\[ E_p = \int_{-\infty}^{\infty} \frac{P_l \cdot P}{4\pi\varepsilon_0 \, (r^2 + \vec{z}'^2)^{3/2}} \, d\vec{z}' \quad \text{from integral table} \]

\[ E_p = \frac{P_l}{4\pi\varepsilon_0} \left( \frac{1}{r^2} - \frac{\vec{z}'}{\sqrt{r^2 + \vec{z}'^2}} \right)_{-\infty}^{+\infty} = \frac{P_l}{2\pi\varepsilon_0 \, P} \]

Electric field intensity due to various charge densities:

1. Point charge \( E(r) \sim \frac{1}{r^2} \)

2. Line charge \( E(P) \sim \frac{1}{P} \)

3. Sheet of charge \( E \) is independent of \( r \)

(see book p. 47-45)

We will see later that intensity of light is proportional to \( E^2 \)

\[ \Rightarrow \text{for uniform lighting that is independent of distance in intensity, sheet sources are used} \]
Sketches of Fields

e.g. field of line charge as viewed along \( z \)-axis

\[ \rightarrow \]

\[ q \]

\[ \rightarrow \]

(length indicates magnitude of field)

'Stream lines' or 'flux lines'

A small positive test charge placed in this field would be accelerated in the direction of the flux lines passing through this point.

Other example: + and - charge

Electric flux density

Imagine these field lines correspond to an electric flux.

The higher the charge, the more flux lines and the higher the flux.
Fara day experiment

- charge inner sphere with charge +Q
- enclose in outer sphere
- discharge outer sphere

≈) experiment shows that charge on outer sphere is then −Q

Displacement or 'flux' from inner sphere to outer:

\[ \Phi = Q \] in Coulombs

Definition of the electric flux density \( \mathbf{D} \):

- magnitude = number of flux lines crossing a surface normal to the lines per surface area
- direction = direction of the flux lines at this point

\[ \mathbf{D}(r=a) = \frac{Q}{4\pi a^2} \mathbf{a}_r \quad (\text{inner sphere}) \]
\[ \mathbf{D}(r=b) = \frac{Q}{4\pi b^2} \mathbf{a}_r \quad (\text{outer sphere}) \]

For \( a \leq r \leq b \)
\[ \mathbf{D}(r) = \frac{Q}{4\pi r^2} \mathbf{a}_r \]

For \( r > b \)
\[ \mathbf{D} = 0 \] , no lines outside of outer sphere!
Electric flux density is measured in $C/m^2$ or "lines per square meter".

(For simplicity we assume that each Coulomb of charge has one electric field line pointing away from it per $m^2$)

The electric field lines terminate on some equal and opposite charge.

Compare with \[ E = \frac{Q}{4\pi \varepsilon_0 r^2} \frac{a}{r} \]

\[ \Rightarrow \quad D = \varepsilon_0 E \quad \text{in free space} \]

\[ D(r) = \frac{Q}{4\pi \varepsilon_0 r^2} \frac{a}{r} \quad \text{is also applicable if charge is embedded in dielectric} \]

\[ Q = D(r) \cdot 4\pi r^2 = \Psi \]

\[ \text{Charge} = \text{Flux density} \times \text{Area} = \text{Flux} \]

units: \[ C = \frac{C}{m^2} \quad m^2 = C \]

Total flux = total charge contained inside that closed surface.
Gauss's law

"Electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

Elemental surface element \( \Delta S \)

\[ \Delta S = \vec{n} \text{ vector normal to surface} \]

\[ \hat{\mathbf{n}} \text{ vector tangential to surface} \]

The elemental surface has both a magnitude and direction.
The direction is given by a unit vector \( \hat{n} \) normal to the surface.

Flux crossing \( \Delta S \):

\[
\Delta \Phi = \vec{D}_s \cdot \hat{n} \cdot \Delta S = \vec{D}_s \cdot \cos(\theta) \Delta S = \vec{D}_s \cdot \Delta S
\]

Total flux:

\[
\Phi = \lim_{\Delta S \to 0} \sum \vec{D}_s \cdot \Delta S
\]

\[
= \int \vec{D}_s \cdot dS
\]

closed surface
\[ \psi = \oint \mathbf{D}_s \cdot d\mathbf{s} = Q = \text{charge enclosed} \]

\[ Q = \int \mathbf{P}_v \, dV = \int \mathbf{P}_l \, dl = \int \mathbf{P}_s \, ds \]

as long as the closed surface contains \( V, S \) or \( L \)

Thus

\[ \oint \mathbf{D}_s \, ds = \int \mathbf{P}_v \, dV \quad \text{Gauss's law} \]

(= one of the four Maxwell equations)

Example: charge \( q \) placed at origin

\[ d\mathbf{s} = r^2 \sin(\theta) \, d\theta \, d\phi \cdot 2\pi \quad \text{(spherical coordinates)} \]

Total flux through closed spherical surface

\[ \Delta \psi = \mathbf{D}_s \cdot d\mathbf{s} \]

\[ \psi = \oint \mathbf{D}_s \, ds = \oint \frac{q}{4\pi \epsilon_0 r^2} \, r \cdot 2\pi \sin(\theta) \, d\theta \, d\phi \cdot 2\pi \]

\[ = \oint d\phi \, \int_0^\pi (\frac{q}{4\pi \epsilon_0} \sin(\theta)) \]

\[ = \frac{q}{4\pi \epsilon_0} \left[ \int_0^{2\pi} d\phi \cdot \int_0^\pi \sin(\theta) \, d\theta \right] \cdot \frac{1}{2\pi} \cdot \frac{1}{2\pi} \cdot \int_0^{2\pi} d\phi \cdot \int_0^\pi \sin(\theta) \, d\theta \cdot \sin(\theta) \]

\[ = \frac{q}{4\pi \epsilon_0} \left[ 2\pi \int_0^\pi \sin(\theta) \, d\theta \right] \]

\[ = \frac{q}{4\pi \epsilon_0} \left[ -\cos(\theta) \right]_0^{\pi} = 2 \]

\[ \Rightarrow \psi = q \]
Applications of Gauss's law:

**Example 1:** point charge $Q$ at origin of spherical coordinates

$$Q = \oint_{\text{sphere}} \mathbf{D}_s \cdot d\mathbf{S}$$

Because $\mathbf{D}_s$ and $d\mathbf{S}$ are always pointed in same direction

$$= D_s \cdot \oint_{\text{sphere}} d\mathbf{S} = D_s \cdot 4\pi r^2$$

$$\Rightarrow D_s = \frac{Q}{4\pi r^2}$$

and thus $\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2} \mathbf{a}_r$ and $E = \frac{Q}{4\pi \epsilon_0 r^2} q r$

**Example 2:** uniform line charge density along $z$

We know that $|E| = |E(\mathbf{r})|$ only

$|\mathbf{D}| = |\mathbf{D}(\mathbf{r})|$ only

(symmetrical around $z$-axis)
We choose a cylinder as closed surface

\[ Q = \oint \mathbf{D}_s \cdot d\mathbf{s} \]

\[ s = \text{cylinder} \]

\[ Q = \oint \mathbf{D}_s \cdot d\mathbf{s} + \oint \mathbf{D}_s \cdot d\mathbf{s} + \oint \mathbf{D}_s \cdot d\mathbf{s} \]

side \quad top \quad bottom

\[ = D_s \oint s \cdot dS + 0 \cdot \oint s \cdot dS + 0 \cdot \oint s \cdot dS \]

assume \( D_s \) to be constant over surface

\[ Q = D_s \oint \frac{L}{2\pi} \int_{\phi=0}^{2\pi} d\phi = D_s \cdot 2\pi P \]

and thus \( D_s = \frac{A}{2\pi P} \)

\[ \mathbf{D}_s = \frac{P_L}{2\pi P} \]

\[ \mathbf{E}_p = \frac{P_L}{2\pi \varepsilon_0 P} \]

Same as result from Coulomb's law

Example 3: coaxial cable

\[ ds = \rho d\phi dz \]

Symmetry: only \( D_s \) present

Choose Gaussian surface: cylinder with \( a < \rho < b \)
\[ Q = \oint D_s \cdot dS = \oint \frac{A}{2\pi R^2} \cdot d\phi \cdot dz \cdot dL = D_s \cdot \frac{2\pi R^2 \cdot L}{2\pi R} = \frac{2\pi a L}{2\pi R} \]

\[ = \frac{a \cdot Ps}{P} \]

The total charge on the inner conductor is

\[ \alpha = Ps \cdot 2\pi a \cdot L \]

\[ D_s = \frac{Q}{2\pi R L} \]

\[ = \frac{Ps}{2\pi R L} \]

\[ = \frac{a \cdot Ps}{P} \]

or

\[ D_s = \frac{a \cdot Ps}{P} \]

\[ P \] outside the outer cylinder?

All electric field lines that start on the inner cylinder terminate on the outer cylinder.

\[ Q_{\text{out}} = -Q_{\text{in}} \]

For \( P > b \):

\[ \oint D_s \cdot dS = Q_{\text{outer}} - Q_{\text{inner}} = 0 \]

\[ \Rightarrow D_s = 0 \]