Theorem: The optimal decision regions under the Bayes Criterion (minimum average cost) are given by a likelihood ratio test (LRT), where

\[ R_0 = \{ x : \Lambda(x) < \Lambda_0 \}, \quad R_1 = \{ x : \Lambda(x) \geq \Lambda_0 \}, \]

and

\[ \Lambda_0 = \frac{\pi_0(C_{10} - C_{00})}{(1 - \pi_0)(C_{01} - C_{11})}. \]

Proof: Consider a different partition of \( \mathbb{R} \), such that \( R'_0 \cup R'_1 = \mathbb{R}, \ R'_0 \cap R'_1 = \emptyset, \) and \( R'_i \neq R_i, \ i = 0, 1. \) Let \( \bar{C} \) be the Bayes cost, and let \( \bar{C}' \) be the cost associated with the new partition. We want to prove that \( \bar{C}' \geq \bar{C}. \)

We have that

\[
\bar{C} = \pi_0 \left[ C_{00} \int_{R_0} p_0(x) \, dx + C_{10} \int_{R_1} p_0(x) \, dx \right] + (1 - \pi_0) \left[ C_{01} \int_{R_0} p_1(x) \, dx + C_{11} \int_{R_1} p_1(x) \, dx \right],
\]

\[
\bar{C}' = \pi_0 \left[ C_{00} \int_{R'_0} p_0(x) \, dx + C_{10} \int_{R'_1} p_0(x) \, dx \right] + (1 - \pi_0) \left[ C_{01} \int_{R'_0} p_1(x) \, dx + C_{11} \int_{R'_1} p_1(x) \, dx \right].
\]

\[
\bar{C}' - \bar{C} = \pi_0 \left[ C_{00} \left( \int_{R'_0} - \int_{R_0} \right) p_0 + C_{10} \left( \int_{R'_1} - \int_{R_1} \right) p_0 \right] + (1 - \pi_0) \left[ C_{01} \left( \int_{R'_0} - \int_{R_0} \right) p_1 + C_{11} \left( \int_{R'_1} - \int_{R_1} \right) p_1 \right].
\]

From

\[
R'_0 = (R'_0 \cap R_0) \cup \underbrace{(R'_0 \cap R_1)}_{r_1}, \quad R'_1 = (\underbrace{R'_1 \cap R_0)}_{r_0} \cup (R'_1 \cap R_1),
\]

\[
R_0 = (R_0 \cap R'_0) \cup \underbrace{(R_0 \cap R_1)}_{r_1}, \quad R_1 = (\underbrace{R_1 \cap R'_0)}_{r_0} \cup (R_1 \cap R'_1),
\]

it follows that

\[
\int_{R'_0} = \int_{R'_0 \cap R_0} + \int_{r_1}, \quad \int_{R'_1} = \int_{R'_1 \cap R_0} + \int_{r_1},
\]

\[
\int_{R_0} = \int_{R_0 \cap R_0} + \int_{r_0}, \quad \int_{R_1} = \int_{R_1 \cap R_0} + \int_{r_1},
\]

\[
\int_{R'_0} - \int_{R_0} = \int_{r_1} - \int_{r_0}, \quad \int_{R'_1} - \int_{R_1} = \int_{r_0} - \int_{r_1}.
\]

Therefore,

\[
\bar{C}' - \bar{C} = \pi_0 (C_{10} - C_{00}) \left[ \left( \int_{r_0} - \int_{r_1} \right) p_0 \right] + (1 - \pi_0) (C_{01} - C_{11}) \left[ \left( \int_{r_0} - \int_{r_1} \right) p_1 \right]
\]

\[
= (1 - \pi_0) (C_{01} - C_{11}) \left[ \int_{r_0} (\Lambda_0 p_0 - p_1) + \int_{r_1} (p_1 - \Lambda_0 p_0) \right].
\]

By definition,

\[
\text{if } x \in r_1 \Rightarrow x \in R_1 \quad \text{and} \quad p_1(x) \geq \Lambda_0 p_0(x), \quad \text{if } x \in r_0 \Rightarrow x \in R_0 \quad \text{and} \quad p_1(x) < \Lambda_0 p_0(x).
\]

In conclusion, \( \bar{C}' - \bar{C} \geq 0 \text{ for any partition, and the Bayes cost is the minimum cost}. \)
**Theorem:** The optimal decision regions under the Neyman-Pearson Criterion (max \( P_D \), for fixed \( P_{FA} = \alpha \)) are given by a likelihood ratio test (LRT), where

\[
\mathcal{R}_0 = \{ x : \Lambda(x) < \Lambda_0 \}, \quad \mathcal{R}_1 = \{ x : \Lambda(x) \geq \Lambda_0 \},
\]

and \( \Lambda_0 \) is chosen so that

\[
\int_{\mathcal{R}_1} p_0(x) \, dx = \alpha.
\]

**Proof:** Consider a different partition of \( \mathbb{R} \), such that \( \mathcal{R}'_0 \cup \mathcal{R}'_1 = \mathbb{R} \), \( \mathcal{R}'_0 \cap \mathcal{R}'_1 = \emptyset \), and \( \mathcal{R}'_i \neq \mathcal{R}_i, \ i = 0, 1 \). By assumption

\[
P_{FA} = \int_{\mathcal{R}_1} p_0(x) \, dx = \int_{\mathcal{R}'_1} p_0(x) \, dx = \alpha.
\]

Let \( P_D \) be the probability of detection associated to the LRT regions \( \mathcal{R}_0 \) and \( \mathcal{R}_1 \) and let \( P'_D \) be the probability of detection under the new partition. We want to prove that \( P'_D \leq P_D \).

\[
P_D = \int_{\mathcal{R}_1} p_1(x) \, dx = \int_{(\mathcal{R}_1 \cap \mathcal{R}'_1) \cup (\mathcal{R}_1 \cap \mathcal{R}'_0)} p_1(x) \, dx
\]

\[
= \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_1(x) \, dx + \int_{\mathcal{R}_1 \cap \mathcal{R}'_0} p_1(x) \, dx
\]

\[
\geq \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_1(x) \, dx + \Lambda_0 \int_{\mathcal{R}_1 \cap \mathcal{R}'_0} p_0(x) \, dx \quad \{ x \in \mathcal{R}_1 \Rightarrow p_1(x) \geq \Lambda_0 p_0(x) \} \}
\]

\[
= \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_1(x) \, dx + \Lambda_0 \left( \int_{\mathcal{R}_1} p_0(x) \, dx - \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_0(x) \, dx \right) \quad \{ \mathcal{R}_1 \cap \mathcal{R}'_0 = \mathcal{R}_1 - \mathcal{R}_1 \cap \mathcal{R}'_1 \}
\]

\[
= \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_1(x) \, dx + \Lambda_0 \left( \int_{\mathcal{R}_1} p_0(x) \, dx - \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_0(x) \, dx \right) \quad \{ \text{by assumption} \}
\]

\[
= \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_1(x) \, dx + \int_{\mathcal{R}_1 \cap \mathcal{R}_0} \Lambda_0 p_0(x) \, dx \quad \{ \mathcal{R}_1' - \mathcal{R}_1' \cap \mathcal{R}_1 = \mathcal{R}_1' \cap \mathcal{R}_0 \}
\]

\[
\geq \int_{\mathcal{R}_1 \cap \mathcal{R}'_1} p_1(x) \, dx + \int_{\mathcal{R}_1 \cap \mathcal{R}_0} p_1(x) \, dx \quad \{ x \in \mathcal{R}_0 \Rightarrow \Lambda_0 p_0(x) > p_1(x) \}
\]

\[
= \int_{\mathcal{R}'_1} p_1(x) \, dx = P'_D.
\]