1. In the frequency domain we have the requirements that

\[ 20 \log_{10} |H(e^{j\omega})| \geq -0.5 \text{ for } 0 \leq \omega \leq 0.2\pi \]

\[ 20 \log_{10} |H(e^{j\omega})| \leq -30 \text{ for } 0.4\pi \leq \omega \leq \pi \]

From \( 20 \log_{10} \sqrt{1 + \epsilon^2} = 0.5 \) dB we get:

\[ \epsilon = \sqrt{10^{0.05} - 1} = 0.34931 \]

And \( 20 \log_{10} A = 30 \) dB, so

\[ A = 10^{1.5} = 31.623 \]

\[ \eta = \frac{\epsilon}{\sqrt{A^2 - 1}} = 0.01105 \]

\[ k = \frac{\tan(\omega_p/2)}{\tan(\omega_s/2)} = \frac{\tan(0.1\pi)}{\tan(0.2\pi)} = 0.44721 \]

\[ N \approx \frac{2}{\pi^2} \ln \left( \frac{4A}{\epsilon} \right) \ln \left( \frac{8}{\frac{1}{k} - 1} \right) = 2.2297 \]

or using Fig. 4a on page 193 in *IEEE Digital Signal Processing II*, we obtain \( N = 3 \).

Wp=0.2
Ws=0.4
Rp=0.5
As=30
\([N,Wn] = \text{ellipord}(Wp,Ws,Rp,As)\)
\([b,a] = \text{ellip}(N,Rp,As,Wn)\)
\(zplane(b,a)\)

Using the above matlab program we find

\[ H(z) = \frac{0.051952(1 + z^{-1})(1 - 0.6642z^{-1} + z^{-2})}{(1 - 0.6299z^{-1})(1 - 1.3588z^{-1} + 0.7339z^{-2})} \]

The digital filter implementation is shown below:

\[ X = [ 7 \ 1 \ 7 ] \]
\[ 1 \ 2 \ 0.051952 \]
\[ 2 \ 3 \ 0.0 \]
\[ 2 \ 4 \ 1.0 \]
\[ 3 \ 2 \ 0.6299 \]
\[ 3 \ 4 \ 1.0 \]
\[ 4 \ 5 \ 0.0 \]
2.

Problem 8.9

\[ H_d(e^{j\omega}) = j\omega \]

\[ h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega \]

\[ h_d(n) = \frac{\omega e^{j\omega n}}{2\pi n} \bigg|_{-\pi}^{\pi} - \frac{1}{2\pi n} \int_{-\pi}^{\pi} e^{j\omega n} d\omega \]

\[ h_d(n) = \frac{\pi e^{j\pi n} + \pi e^{-j\pi n}}{2\pi n} - \frac{e^{j\pi n}}{2\pi n^2} \bigg|_{\pi} \]

\[ h_d(n) = \left( \frac{\cos(\pi n)}{n} - \frac{\sin(\pi n)}{\pi n^2} \right) \]

So if \( n \neq 0 \) then \( h_d(n) = \frac{\cos(\pi n)}{n} \) and if \( n = 0 \) then by l’ Hopital we find

\[ h_d(0) = \lim_{n\to0} \frac{\pi n \cos(\pi n) - \sin(\pi n)}{\pi n^2} = \lim_{n\to0} \frac{-\pi^2 n \sin(\pi n)}{2\pi n} = 0 \]

Using a FIR filter with window length \( M = 21 \) we get

\[ h_{d,causal}(n) = h_d(n - 10) = \left\{ \begin{array}{ll}
\frac{\cos(\pi(n-10))}{n-10} & \text{for } 0 \leq n \leq 20, \ n \neq 10 \\
0 & \text{for } n < 0, \ n = 10, \ n > 20
\end{array} \right. \]

The Hamming window has an impulse response

\[ w(n) = 0.54 + 0.46 \cos \left( \frac{2\pi(n - 10)}{20} \right) \text{ for } 0 \leq n \leq 20 \]

And \( h_{actual}(n) = h_{d,causal}(n) \cdot w(n) \).

With a Hamming window we obtain the following frequency response.
Problem 8.15

In the frequency domain we have the requirements that

\[ 20 \log_{10} |H(e^{j\omega})| \geq 1 \text{ for } 0 \leq \omega \leq \pi/3 \]

\[ 20 \log_{10} |H(e^{j\omega})| \leq -40 \text{ for } \pi/2 \leq \omega \leq \pi \]

From \( 20 \log_{10} \sqrt{1 + \epsilon^2} = 1 \text{dB} \) we get:

\[ \epsilon = \sqrt{10^{0.1} - 1} = 0.50885 \]

And \( 20 \log_{10} A = 40 \text{dB} \), so

\[ A = 10^2 = 100 \]

\[ \eta = \frac{\epsilon}{\sqrt{A^2 - 1}} = 5.0888 \times 10^{-3} \]

\[ k = \frac{\tan(\omega_p/2)}{\tan(\omega_s/2)} = \frac{\tan(\pi/6)}{\tan(\pi/4)} = \frac{1}{\sqrt{3}} = 0.57735 \]

Butterworth filter:

\[ N = \frac{\log(\eta)}{\log(k)} = 9.6134 \rightarrow 10 \]

Chebyshev filter:

\[ N = \frac{\log \left( \frac{1 + \sqrt{1 - \eta^2}}{\eta} \right)}{\log \left( \frac{1 + \sqrt{1 - k^2}}{k} \right)} = 5.2118 \rightarrow 6 \]

Elliptic filter:

\[ N \approx \frac{2}{\pi^2} \ln \left( \frac{4A}{\epsilon} \right) \ln \left( \frac{8}{1 - k} \right) = 3.23 \rightarrow 4 \]
3rd-order Elliptic low-pass filter
Cascade-form 3th-order Elliptic low-pass filter

passband
Problem 8.9: Differentiator with Hamming window

abs(h)

omega/\pi

Phase

degree

omega/\pi
specifications can be found using the following relations:

\[
k = \frac{\Omega_p}{\Omega_s},
\]

\[
k_1 = \frac{\epsilon}{\sqrt{A^2 - 1}},
\]

\[
N \geq \frac{K(k)K(\sqrt{1 - k_1^2})}{K(k_1)K(\sqrt{1 - k^2})}, \tag{5.22}
\]

where \(k\) is known as the transition ratio and \(K(\cdot)\) is the complete elliptic integral of the first kind [AB65]:

\[
K(k) = \int_0^{\pi/2} \frac{d\phi}{(1 + k^2 \sin^2 \phi)^{1/2}}
\]

\[
= \frac{\pi}{2} \left( 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \cdots \right)
\]

\[
= \text{sn}^{-1}(1, k),
\]

where if we let \(\text{sn}(u, k)\) be the Jacobian elliptic function and \(y = \text{sn}(u, k)\), then

\[
\text{sn}^{-1}(y, k) = u = \int_0^y \frac{dt}{(1 - t^2)^{1/2} (1 - k^2 t^2)^{1/2}}.
\]

In lieu of Eq. (5.22), the approximation

\[
N = \frac{2}{\pi^2} \ln \left( \frac{4A}{\epsilon} \right) \ln \left[ \frac{\Omega_p}{(\Omega_s - \Omega_p)} \right] \tag{5.23}
\]

is useful in estimating the filter order [ST57, GR76].

The poles \(s_i = \alpha_i + j\Omega_s\), \(i = 0, 1, \ldots, N - 1\), of \(H_e(s)\) are specified by the relations [GO69, ST57]

\[
\sigma_i = \Omega_p \frac{\text{sn}(q, k') \text{cn}(q, k') \text{cn}(r_j, k) \text{dn}(r_j, k)}{1 - \text{sn}^2(q, k') \text{dn}^2(r_j, k)},
\]

\[
\Omega_i = \Omega_p \frac{\text{sn}(r_j, k) \text{dn}(q, k')}{1 - \text{sn}^2(q, k') \text{dn}^2(r_j, k)},
\]

\[
\text{cn}^2(u, k) = 1 - \text{sn}^2(u, k),
\]

\[
\text{dn}^2(u, k) = 1 - k^2 \text{sn}^2(u, k),
\]

\[
k' = \sqrt{1 - k^2},
\]