Midterm Solutions

Note: For all problems please circle or otherwise clearly indicate your answers!

1. **2D Convolution and Autocorrelation (25 points)**

Consider the continuous 2D function \( f(x,y) \) given below, which has value 1 where the dark regions are located and 0 elsewhere.

(a) (15 points) Make a sketch showing the boundaries of non-zero regions of the function \( g(x,y) \), where \( g(x,y) = f(x,y) \ast f(x,y) \). Be sure to place your sketch on a set of marked axis such that relevant dimensions and positions of features of \( g(x,y) \) are clearly indicated.

Answer:
(b) (10 points) Make a sketch showing the boundaries of non-zero regions of the autocorrelation of \( f(x,y) \). Again, your answer must be clearly indicated for you to receive credit.

Answer:

2. 2D Linear Systems (20 points)

For the following 2D difference equation:

\[
y(m, n) = y(m-1, n-2) + y(m+1, n) + x(m+1, n-1)
\]

Find and plot the region of support of the impulse response of the 2D system. Your plot must include the points in the square, origin-centered region \(-6 \leq m, n \leq 6\)
3. DFT (30 points)

a) (10 points) Consider a sequence \( u(n) \) which is periodic with period 4. Let \( u(n) \) be represented over the range \( 0 \leq n \leq 3 \) by:

\[
u(n) = [a \ b \ c \ d]
\]

The underline under the \( a \) indicates that it is at the origin; e.g. that it occupies the \( n=0 \) position. Let \( w(n) = [A \ B \ C \ D] = u(-n) \). Specify each of the elements of \( w(n) \), \( A, B, C, \) and \( D \) in terms of \( a, b, c, \) and \( d \).

Answer:

\[
w(n) = [a \ d \ c \ b], \text{ meaning that } A = a, B = d, C = c, \text{ and } D = b.
\]

b) (5 points) Suppose that \( u(n) \) is known to be odd. Specify completely all dependencies that will exist among \( a, b, c, \) and \( d \).

Answer:

\[
a = c = 0
\]

\[b = -d
\]

b) (5 points) Suppose that \( u(n) \) is known to be odd. Specify completely all dependencies that will exist among \( a, b, c, \) and \( d \).

Answer:

\[
a = c = 0
\]

\[b = -d
\]

c) (15 points) Consider the DFT for \( N=4 \). Such a DFT relates \( u(n) \) with \( v(k) \), where, in this case, both \( u(n) \) and \( v(k) \) have period 4.

Provide an example of specific values for \( a, b, c, \) and \( d \) for a real, even sequence \( u(n) \) that is self-transforming; in other words that satisfies \( u(n) = v(k) = [a \ b \ c \ d] \). Hint: For this problem, assume that \( b = -d \).
Answer:

First it is important to realize that the Fourier transform implies the following relations:

\[
\begin{align*}
  a &= \frac{1}{2} \left[ a + b + c + d \right] \\
  b &= \frac{1}{2} \left[ a - bj - c + dj \right] \\
  c &= \frac{1}{2} \left[ a - b + c - d \right] \\
  d &= \frac{1}{2} \left[ a + bj - c - dj \right]
\end{align*}
\]

Looking at the first equation:

\[
 a = \frac{1}{2} \left[ a + b + c + d \right]
\]

Now we use the hint that \( d = -b \).

\[
 a = \frac{1}{2} \left[ a + b + c - b \right] = \frac{1}{2} \left[ a + c \right]
\]

Now looking at the second equation:

\[
 b = \frac{1}{2} \left[ a - bj - c + dj \right]
\]

we use the relation \( d = -b \)

\[
 b = \frac{1}{2} \left[ a - bj - c - bj \right] = \frac{1}{2} \left[ a - 2b \right]
\]

\[
 b = \frac{a - c}{2 + 2j}
\]

but if we use the relation we just found, that \( a = c \), then

\[
 b = \frac{a - a}{2 + 2j} = 0
\]

This completes the answer, \( a = c \) and \( b = -d = 0 \)

So the answer should be in the form

\[
\begin{bmatrix}
  a & b & c & d
\end{bmatrix} = \begin{bmatrix}
  x & 0 & x & 0
\end{bmatrix}
\]

Where \( x \) is any number.
4. Image transforms (25 points)

The figure below is a 256 by 256 image $u(m,n)$ consisting of a vertical line. This line passes through the origin, which is at the center of the image below. The coordinates of the four corners are as indicated. The line is one pixel thick and is composed of pixels with a value of 255.

Provide a complete description of the transform, $v(k,l)$. Your description should specify, for every value $(k,l)$ in the range $0 \leq k,l \leq 255$, (or, equivalently, $-127 \leq k,l \leq 128$) the exact value of $v(k,l)$.

![Image of a 256 by 256 image with a vertical line](image)

Answer:

$$v(k,l) = \frac{1}{256} \sum_{m=0}^{255} \sum_{n=0}^{255} u(m,n) e^{-j2\pi(mk+nl)/N}$$

Since $u(m,n)$ is 255 when $n=0$, and zero otherwise

$$v(k,l) = \frac{1}{256} \sum_{m=0}^{255} 255 \cdot e^{j2\pi mk/256} \text{ note } v(k,l) \text{ depends on } k \text{ only}$$

Thus $v(k)$ is the transform of a 1D function, that is a DC value of 255. As we have seen
When \( k=0 \), the sum becomes:

\[
\sum_{m=0}^{255} 255 = \frac{256 \cdot 255}{256} = 255
\]

When \( k \neq 0 \), we can use:

\[
\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}
\]

giving:

\[
\sum_{n=0}^{N-1} a^n = \frac{1-e^{-j2\pi k 256}}{1-e^{-j2\pi k}} = 0
\]

So \( v(k,l) \) is 255 when \( k=0 \), 0 otherwise, and has no dependence on \( l \). This is an image with a line perpendicular to input, as shown below.