Midterm Solution

Note: For all problems please circle or otherwise clearly indicate your answers!

1. 2D Convolution and Autocorrelation (25 points)

Consider the continuous 2D function \( f(x,y) \) given below, which has value 1 where the dark regions are located and 0 elsewhere.

(a) (15 points) Make a sketch showing the boundaries of non-zero regions of the function \( g(x,y) \), where \( g(x,y) = f(x,y) * f(x,y) \). Be sure to place your sketch on a set of marked axis such that relevant dimensions and positions of features of \( g(x,y) \) are clearly indicated.
(b) (10 points) Make a sketch showing the boundaries of non-zero regions of the autocorrelation of \( f(x,y) \). Again, your answer must be clearly indicated for you to receive credit.

2. **2D Linear Systems (20 points)**

For the following 2D difference equation:

\[
y(m,n) = y(m-1, n-2) + y(m, n-1) + x(m, n+2) + x(m+1, n+1)
\]

(a) (5 points) Find and plot the output mask of this 2D system.
(b) (15 points) Find and plot the region of support of the impulse response of the 2D system. Your plot must include the points in the square, origin-centered region $-6 < m, n < 6$

3. One dimensional DFT (25 points)

Determine whether it is possible to find four (possible complex) constants $a, b, c$ and $d$ such that the following constitutes a valid unitary N=4 DFT pair:

$$x(n) = \{-1, 2, a, b\} \longleftrightarrow \frac{1}{2} \{5 + j, -1 + j, c, d\} = X(k)$$

If a unique solution is possible, find the four constants $a, b, c,$ and $d$. If a solution is not possible, prove why not.

$$[-1, 2, a, b] \leftrightarrow \begin{bmatrix} \frac{-1}{2}, & \frac{-1}{2}, & \frac{-1}{2}, & \frac{-1}{2} \\ \end{bmatrix} + \begin{bmatrix} 1, & -j, & -1, & j \\ \end{bmatrix}$$

$$+ \begin{bmatrix} a, & -a, & a, & -a \\ \frac{2}{2}, & \frac{-a}{2}, & \frac{a}{2}, & \frac{-a}{2} \\ \end{bmatrix}$$

$$+ \begin{bmatrix} b, & -j b, & b, & -j b \\ \frac{2}{2}, & \frac{-j b}{2}, & \frac{b}{2}, & \frac{-j b}{2} \\ \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{a}{2} + \frac{b}{2}, & \frac{-1}{2} - j - \frac{a}{2} + j \frac{b}{2}, & \frac{-3}{2} + \frac{a}{2} - \frac{b}{2}, & \frac{-1}{2} + j - \frac{a}{2} - j \frac{b}{2} \\ \end{bmatrix}$$
Equating this to \( X(k) \) gives:

\[
\frac{1}{2} + \frac{a}{2} + \frac{b}{2} = \frac{1}{2} (5 + j) \quad (1)
\]
\[
-\frac{1}{2} - j - \frac{a}{2} + j \frac{b}{2} = \frac{1}{2} (-1 + j) \quad (2)
\]
\[
-\frac{3}{2} + \frac{a}{2} - \frac{b}{2} = \frac{c}{2} \quad (3)
\]
\[
-\frac{1}{2} + j - \frac{a}{2} - j \frac{b}{2} = \frac{d}{2} \quad (4)
\]

From (1), \( 6 = 4 + j - a \), which when substituted in (2) gives: \( a = j \)

Which then means \( \frac{1}{2} (5 + j) = \frac{1}{2} + \frac{j}{2} + \frac{b}{2} \Rightarrow b = 4 \Rightarrow c = -7 + j \Rightarrow d = -1 - 3j \)

4. **Understanding Transforms (25 points)**

The figure below shows the absolute value (scaled logarithmically as in the computer assignments) of the DFT \( v(k,l) \) of a 256 by 256 image. The transform (before the absolute value operation) was purely real. The only nonzero portions of \( u(m,n) \) are two squares of identical size. Say as much as you can about the size and location of these squares.
This is clearly a separable 2D function. The 'large' scale feature is a sinc function (or something very close) with zero crossings that first occur at \( \pm k = \pm f \equiv 16 \).

This corresponds to an input that is square of size \( \equiv 16 \times 16 \).

You can confirm this by realizing that a \( 16 \times 16 \) square will have elements at most 8 pixels away from the origin \( \Rightarrow \) this will give rise to oscillations with period 16 in the transform, which will become period 8 after the absolute value operation.

The finer oscillations appear only in the vertical direction, and have period (before doubling due to absolute value) of about \( (2/3) \times 16 = 10 \frac{2}{3} \). This corresponds to convolution in the space domain with a pair of delta functions at distance \( 128/(10 \frac{2}{3}) \equiv 12 \) from the origin. So the input is 2 squares, each size \( 16 \times 16 \), displaced by vertical distances of \( \pm 12 \) from the origin.