1.4. A circle, centered at the origin with a radius of 2 units, lies in the $xy$ plane. Determine the unit vector in rectangular components that lies in the $xy$ plane, is tangent to the circle at $(\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of $y$:

A unit vector tangent to this circle in the general increasing $y$ direction is $\mathbf{t} = \mathbf{a}_y$. Its $x$ and $y$ components are $t_x = \mathbf{a}_y \cdot \mathbf{a}_x = -\sin \phi$, and $t_y = \mathbf{a}_y \cdot \mathbf{a}_y = \cos \phi$. At the point $(\sqrt{3}, 1), \phi = 30^\circ$, and so $t = -\sin 30^\circ \mathbf{a}_x + \cos 30^\circ \mathbf{a}_y = 0.5(-\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)$.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find:

a) $\mathbf{G}$ at $P$: $\mathbf{G}(1, 2, -1) = (48, 36, 18)$

b) a unit vector in the direction of $\mathbf{G}$ at $Q$: $\mathbf{G}(-2, 1, 3) = (-48, 72, 162)$, so

$$\mathbf{a}_G = \frac{(-48, 72, 162)}{|(-48, 72, 162)|} = \left(\frac{-0.26, 0.39, 0.88}{\sqrt{162}}\right)$$

c) a unit vector directed from $Q$ toward $P$:

$$\mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3, -1, 4)}{\sqrt{26}} = \left(\frac{0.59, 0.20, -0.78}{\sqrt{26}}\right)$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2 + 2), 18z^2)|$, or $10 = |(4xy, 2x^2 + 4, 3z^2)|$, so the equation is

$$100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4$$

1.6. If $\mathbf{a}$ is a unit vector in a given direction, $B$ is a scalar constant, and $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, describe the surface $\mathbf{r} \cdot \mathbf{a} = B$. What is the relation between the the unit vector $\mathbf{a}$ and the scalar $B$ to this surface? (HINT: Consider first a simple example with $\mathbf{a} = \mathbf{a}_x$ and $B = 1$, and then consider any $\mathbf{a}$ and $B$):

We could consider a general unit vector, $\mathbf{a} = A_1\mathbf{a}_x + A_2\mathbf{a}_y + A_3\mathbf{a}_z$, where $A_1^2 + A_2^2 + A_3^2 = 1$. Then $\mathbf{r} \cdot \mathbf{a} = A_1x + A_2y + A_3z = f(x, y, z) = B$. This is the equation of a planar surface, where $f = B$. The relation of $\mathbf{a}$ to the surface becomes clear in the special case in which $\mathbf{a} = \mathbf{a}_x$. We obtain $\mathbf{r} \cdot \mathbf{a} = f(x) = x = B$, where it is evident that $\mathbf{a}$ is a unit normal vector to the surface (as a look ahead (Chapter 4), note that taking the gradient of $f$ gives $\mathbf{a}$).

1.7. Given the vector field $\mathbf{E} = 4zy^2\cos 2x\mathbf{a}_x + 2zy\sin 2x\mathbf{a}_y + y^2\sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find:

a) the surfaces on which $E_y = 0$. With $E_y = 2zy\sin 2x = 0$, the surfaces are 1) the plane $z = 0$, with $|x| < 2, |y| < 2$; 2) the plane $y = 0$, with $|x| < 2, |z| < 2$; 3) the plane $x = 0$, with $|y| < 2, |z| < 2$; 4) the plane $x = \pi/2$, with $|y| < 2, |z| < 2$.

b) the region in which $E_y = E_z$: This occurs when $2zy\sin 2x = y^2\sin 2x$, or on the plane $2z = y$, with $|x| < 2, |y| < 2, |z| < 1$.

c) the region in which $\mathbf{E} = 0$: We would have $E_x = E_y = E_z = 0$, or $zy^2\cos 2x = zy\sin 2x = y^2\sin 2x = 0$. This condition is met on the plane $y = 0$, with $|x| < 2, |z| < 2$. 

\[2\]
1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $R_{MN}$: $R_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = (-0.3, 0.3, 0.4)$.

b) the dot product $R_{MN} \cdot R_{MP}$: $R_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $R_{MN} \cdot R_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = 0.05$.

c) the scalar projection of $R_{MN}$ on $R_{MP}$:

$$R_{MN} \cdot a_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = 0.12$$

d) the angle between $R_{MN}$ and $R_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{R_{MN} \cdot R_{MP}}{|R_{MN}| |R_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.09 \times 0.34 \times 0.17}}\right) = 78^\circ$$

1.12. Show that the vector fields $A = \rho \cos \phi a_\rho + \rho \sin \phi a_\phi + \rho a_z$ and $B = \rho \cos \phi a_\rho + \rho \sin \phi a_\phi - \rho a_z$ are everywhere perpendicular to each other.

We find $A \cdot B = \rho^2(\sin^2 \phi + \cos^2 \phi) - \rho^2 = 0 = |A||B| \cos \theta$. Therefore $\cos \theta = 0$ or $\theta = 90^\circ$.

1.13. a) Find the vector component of $F = (10, -6, 5)$ that is parallel to $G = (0.1, 0.2, 0.3)$:

$$F_{\parallel G} = \frac{F \cdot G}{|G|^2} G = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09} (0.1, 0.2, 0.3) = (0.93, 1.86, 2.79)$$

b) Find the vector component of $F$ that is perpendicular to $G$:

$$F_{\perp G} = F - F_{\parallel G} = (10, -6, 5) - (0.93, 1.86, 2.79) = (9.07, -7.86, 2.21)$$

c) Find the vector component of $G$ that is perpendicular to $F$:

$$G_{\perp F} = G - G_{\parallel F} = G - \frac{G \cdot F}{|F|^2} F = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25} (10, -6, 5) = (0.02, 0.25, 0.26)$$

1.14. Show that the vector fields $A = a_r (\sin 2\theta)/r^2 + 2a_\theta (\sin \theta)/r^2$ and $B = r \cos \theta a_r + r a_\theta$ are everywhere parallel to each other:

Using the definition of the cross product, we find

$$A \times B = \left(\frac{\sin 2\theta}{r} - \frac{2 \sin \theta \cos \theta}{r}\right) a_\phi = 0 = |A||B| \sin \theta n$$

Identify $n = a_\phi$, and so $\sin \theta = 0$, and therefore $\theta = 0$ (they’re parallel).
1.17b) Find a unit vector in the plane of the triangle and perpendicular to \( \mathbf{R}_{AN} \):

\[
\mathbf{a}_{AN} = \left( -\frac{10}{\sqrt{389}}, 8, 15 \right) = (-0.507, 0.406, 0.761)
\]

Then

\[
\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = (-0.550, -0.832, 0.077)
\]

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at \( A \): A non-unit vector in the required direction is \((1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})\), where

\[
\mathbf{a}_{AM} = \left( \frac{20, 18, -10}{\sqrt{(20, 18, -10)}} \right) = (0.697, 0.627, -0.348)
\]

Now

\[
\frac{1}{2}(\mathbf{a}_{AM} + \mathbf{a}_{AN}) = \frac{1}{2} \left[ (0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761) \right] = (0.095, 0.516, 0.207)
\]

Finally,

\[
\mathbf{a}_{bis} = \frac{(0.095, 0.516, 0.207)}{\left| (0.095, 0.516, 0.207) \right|} = (0.168, 0.915, 0.367)
\]

1.18. Transform the vector field \( \mathbf{H} = (A/\rho) \mathbf{a}_\phi \), where \( A \) is a constant, from cylindrical coordinates to spherical coordinates:

First, the unit vector does not change, since \( \mathbf{a}_\phi \) is common to both coordinate systems. We only need to express the cylindrical radius, \( \rho \), as \( \rho = r \sin \theta \), obtaining

\[
\mathbf{H}(r, \theta) = \frac{A}{r \sin \theta} \mathbf{a}_\phi
\]

1.19. a) Express the field \( \mathbf{D} = (x^2 + y^2)^{-1}(x \mathbf{a}_x + y \mathbf{a}_y) \) in cylindrical components and cylindrical variables:

\[
\text{Have } x = \rho \cos \phi, \ y = \rho \sin \phi, \ \text{and } x^2 + y^2 = \rho^2. \text{ Therefore}
\]

\[
\mathbf{D} = \frac{1}{\rho} (\cos \phi \mathbf{a}_x + \sin \phi \mathbf{a}_y)
\]

Then

\[
D_\rho = \mathbf{D} \cdot \mathbf{a}_\rho = \frac{1}{\rho} \left[ \cos \phi (\mathbf{a}_x \cdot \mathbf{a}_\rho) + \sin \phi (\mathbf{a}_y \cdot \mathbf{a}_\rho) \right] = \frac{1}{\rho} \left[ \cos^2 \phi + \sin^2 \phi \right] = \frac{1}{\rho}
\]

and

\[
D_\phi = \mathbf{D} \cdot \mathbf{a}_\phi = \frac{1}{\rho} \left[ \cos \phi (\mathbf{a}_x \cdot \mathbf{a}_\phi) + \sin \phi (\mathbf{a}_y \cdot \mathbf{a}_\phi) \right] = \frac{1}{\rho} \left[ \cos \phi (-\sin \phi) + \sin \phi \cos \phi \right] = 0
\]

Therefore

\[
\mathbf{D} = \frac{1}{\rho} \mathbf{a}_\rho
\]
1.19b) Evaluate \( \mathbf{D} \) at the point where \( \rho = 2, \phi = 0.2\pi, \) and \( z = 5 \), expressing the result in cylindrical and cartesian coordinates: At the given point, and in cylindrical coordinates, \( \mathbf{D} = 0.5a_\rho \). To express this in cartesian, we use

\[
\mathbf{D} = 0.5(a_\rho \cdot a_x) a_x + 0.5(a_\rho \cdot a_y) a_y = 0.5 \cos 36^\circ a_x + 0.5 \sin 36^\circ a_y = 0.41a_x + 0.29a_y
\]

1.20. A cylinder of radius \( a \), centered on the \( z \) axis, rotates about the \( z \) axis at angular velocity \( \Omega \) rad/s. The rotation direction is counter-clockwise when looking in the positive \( z \) direction.

a) Using cylindrical components, write an expression for the velocity field, \( \mathbf{v} \), that gives the tangential velocity at any point within the cylinder:

Tangential velocity is angular velocity times the perpendicular distance from the rotation axis.

With counter-clockwise rotation, we therefore find \( \mathbf{v}(\rho) = -\Omega \rho a_\phi \) (\( \rho < a \)).

b) Convert your result from part \( a \) to spherical components:

In spherical, the component direction, \( a_\phi \), is the same. We obtain

\[
\mathbf{v}(r, \theta) = -\Omega r \sin \theta a_\phi \ (r \sin \theta < a)
\]

c) Convert to rectangular components:

\[
v_x = -\Omega \rho a_\phi \cdot a_x = -\Omega (x^2 + y^2)^{1/2} (-\sin \phi) = -\Omega (x^2 + y^2)^{1/2} \frac{-y}{(x^2 + y^2)^{1/2}} = \Omega y
\]

Similarly

\[
v_y = -\Omega \rho a_\phi \cdot a_y = -\Omega (x^2 + y^2)^{1/2} (\cos \phi) = -\Omega (x^2 + y^2)^{1/2} \frac{x}{(x^2 + y^2)^{1/2}} = -\Omega x
\]

Finally \( \mathbf{v}(x, y) = \Omega [y a_x - x a_y] \), where \((x^2 + y^2)^{1/2} < a\).

1.21. Express in cylindrical components:

a) the vector from \( C(3, 2, -7) \) to \( D(-1, -4, 2) \):

\[
\begin{align*}
C(3, 2, -7) & \rightarrow C(\rho = 3.61, \phi = 33.7^\circ, z = -7) \text{ and } \\
D(-1, -4, 2) & \rightarrow D(\rho = 4.12, \phi = -104.0^\circ, z = 2) \text{.}
\end{align*}
\]

Now \( \mathbf{R}_{CD} = (-4, -6, 9) \) and \( R_\rho = \mathbf{R}_{CD} \cdot a_\rho = -4 \cos (33.7) - 6 \sin (33.7) = -6.66 \). Then \( R_\phi = \mathbf{R}_{CD} \cdot a_\phi = 4 \sin (33.7) - 6 \cos (33.7) = -2.77 \). So \( \mathbf{R}_{CD} = -6.66 a_\rho - 2.77 a_\phi + 9a_z \)

b) a unit vector at \( D \) directed toward \( C \):

\[
\mathbf{R}_{CD} = (4, 6, -9) \text{ and } R_\rho = \mathbf{R}_{DC} \cdot a_\rho = 4 \cos (-104.0) + 6 \sin (-104.0) = -6.79 \). Then \( R_\phi = \mathbf{R}_{DC} \cdot a_\phi = 4 [-\sin (-104.0)] + 6 \cos (-104.0) = 2.43 \). So \( \mathbf{R}_{DC} = -6.79 a_\rho + 2.43 a_\phi - 9a_z \)

Thus \( a_{DC} = -0.59 a_\rho + 0.21 a_\phi - 0.78 a_z \)

c) a unit vector at \( D \) directed toward the origin: Start with \( \mathbf{r}_D = (-1, -4, 2) \), and so the vector toward the origin will be \( -\mathbf{r}_D = (1, 4, -2) \). Thus in cartesian the unit vector is \( \mathbf{a} = (0.22, 0.87, -0.44) \). Convert to cylindrical:

\[
\begin{align*}
a_\rho &= (0.22, 0.87, -0.44) \cdot a_\rho = 0.22 \cos (-104.0) + 0.87 \sin (-104.0) = -0.90, \text{ and } \\
a_\phi &= (0.22, 0.87, -0.44) \cdot a_\phi = 0.22 [-\sin (-104.0)] + 0.87 \cos (-104.0) = 0, \text{ so that finally, } \\
\mathbf{a} &= -0.90 a_\rho - 0.44 a_z .
\end{align*}
\]
1.22. A sphere of radius $a$, centered at the origin, rotates about the $z$ axis at angular velocity $\Omega$ rad/s. The rotation direction is clockwise when one is looking in the positive $z$ direction.

a) Using spherical components, write an expression for the velocity field, $v$, which gives the tangential velocity at any point within the sphere:

As in problem 1.20, we find the tangential velocity as the product of the angular velocity and the perpendicular distance from the rotation axis. With clockwise rotation, we obtain

$$v(r, \theta) = \Omega r \sin \theta \hat{a}_\phi \quad (r < a)$$

b) Convert to rectangular components:

From here, the problem is the same as part $c$ in Problem 1.20, except the rotation direction is reversed. The answer is $v(x, y) = \Omega [-y \hat{a}_x + x \hat{a}_y]$, where $(x^2 + y^2 + z^2)^{1/2} < a$.

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100^\circ$, $\phi = 130^\circ$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \, dz = 6.28$$

NOTE: The limits on the $\phi$ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2 \int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi + \int_3^{4.5} \int_{100^\circ}^{130^\circ} 3 \, d\phi \, dz$$

$$+ \int_3^{4.5} \int_{100^\circ}^{130^\circ} 5 \, d\phi \, dz + 2 \int_3^5 \int_3^5 \, d\rho \, dz = 20.7$$

c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 + 4 \times 2 + 2 \times \left[ \frac{30^\circ}{360^\circ} \times 2\pi \times 3 + \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = 22.4$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points $A(\rho = 3, \phi = 100^\circ, z = 3)$ and $B(\rho = 5, \phi = 130^\circ, z = 4.5)$. Performing point transformations to cartesian coordinates, these become $A(x = -0.52, y = 2.95, z = 3)$ and $B(x = -3.21, y = 3.83, z = 4.5)$. Taking $A$ and $B$ as vectors directed from the origin, the requested length is

$$\text{Length} = ||\mathbf{B} - \mathbf{A}|| = |(-2.69, 0.88, 1.5)| = 3.21$$
1.24. Express the field $\mathbf{E} = A\mathbf{a}_r/r^2$ in
a) rectangular components:

$$E_x = \frac{A}{r^2} \mathbf{a}_r \cdot \mathbf{a}_x = \frac{A}{r^2} \sin \theta \cos \phi = \frac{A}{x^2 + y^2 + z^2} \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{x}{\sqrt{x^2 + y^2}} = \frac{Ax}{(x^2 + y^2 + z^2)^{3/2}}$$

$$E_y = \frac{A}{r^2} \mathbf{a}_r \cdot \mathbf{a}_y = \frac{A}{r^2} \sin \theta \sin \phi = \frac{A}{x^2 + y^2 + z^2} \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{y}{\sqrt{x^2 + y^2}} = \frac{Ay}{(x^2 + y^2 + z^2)^{3/2}}$$

$$E_z = \frac{A}{r^2} \mathbf{a}_r \cdot \mathbf{a}_z = \frac{A}{r^2} \cos \theta = \frac{A}{x^2 + y^2 + z^2} \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{Az}{(x^2 + y^2 + z^2)^{3/2}}$$

Finally

$$\mathbf{E}(x, y, z) = \frac{A(x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z)}{(x^2 + y^2 + z^2)^{3/2}}$$

b) cylindrical components: First, there is no $\mathbf{a}_\phi$ component, since there is none in the spherical representation. What remains are:

$$E_\rho = \frac{A}{r^2} \mathbf{a}_r \cdot \mathbf{a}_\rho = \frac{A}{r^2} \sin \theta = \frac{A}{(\rho^2 + z^2)} \frac{\rho}{\sqrt{\rho^2 + z^2}} = \frac{A\rho}{(\rho^2 + z^2)^{3/2}}$$

and

$$E_z = \frac{A}{r^2} \mathbf{a}_r \cdot \mathbf{a}_z = \frac{A}{r^2} \cos \theta = \frac{A}{(\rho^2 + z^2)} \frac{z}{\sqrt{\rho^2 + z^2}} = \frac{Az}{(\rho^2 + z^2)^{3/2}}$$

Finally

$$\mathbf{E}(\rho, z) = \frac{A(\rho \mathbf{a}_\rho + z \mathbf{a}_z)}{(\rho^2 + z^2)^{3/2}}$$

1.25. Given point $P(r = 0.8, \theta = 30^\circ, \phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2} \left( \cos \phi \mathbf{a}_r + \frac{\sin \phi}{\sin \theta} \mathbf{a}_\phi \right)$$

a) Find $\mathbf{E}$ at $P$: $\mathbf{E} = 1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi$.

b) Find $|\mathbf{E}|$ at $P$: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = 2.47$.

c) Find a unit vector in the direction of $\mathbf{E}$ at $P$:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = 0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi$$

1.26. Express the uniform vector field, $\mathbf{F} = 5\mathbf{a}_x$, in
a) cylindrical components: $F_\rho = 5 \mathbf{a}_x \cdot \mathbf{a}_\rho = 5 \cos \phi$, and $F_\phi = 5 \mathbf{a}_x \cdot \mathbf{a}_\phi = -5 \sin \phi$. Combining, we obtain $\mathbf{F}(\rho, \phi) = 5(\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi)$.

b) spherical components: $F_r = 5 \mathbf{a}_x \cdot \mathbf{a}_r = 5 \sin \theta \cos \phi$; $F_\theta = 5 \mathbf{a}_x \cdot \mathbf{a}_\theta = 5 \cos \theta \cos \phi$; $F_\phi = 5 \mathbf{a}_x \cdot \mathbf{a}_\phi = -5 \sin \phi$. Combining, we obtain $\mathbf{F}(r, \theta, \phi) = 5[\sin \theta \cos \phi \mathbf{a}_r + \cos \theta \cos \phi \mathbf{a}_\theta - \sin \phi \mathbf{a}_\phi]$. 

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1.28a) (continued)

\[ G_z = 8 \sin \phi \mathbf{a}_\theta \cdot \mathbf{a}_z = 8 \sin \phi (-\sin \theta) = \frac{-8y}{\sqrt{x^2 + y^2}} \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \]

\[ = \frac{-8y}{\sqrt{x^2 + y^2 + z^2}} \]

Finally,

\[ \mathbf{G}(x, y, z) = \frac{8y}{\sqrt{x^2 + y^2 + z^2}} \left[ \frac{xz}{x^2 + y^2} \mathbf{a}_x + \frac{yz}{x^2 + y^2} \mathbf{a}_y - \mathbf{a}_z \right] \]

b) cylindrical components: The \( \mathbf{a}_\theta \) direction will transform to cylindrical components in the \( \mathbf{a}_r \) and \( \mathbf{a}_z \) directions only, where

\[ G_\rho = 8 \sin \phi \mathbf{a}_\theta \cdot \mathbf{a}_\rho = 8 \sin \phi \cos \theta = 8 \sin \phi \frac{z}{\sqrt{\rho^2 + z^2}} \]

The \( z \) component will be the same as found in part \( a \), so we finally obtain

\[ \mathbf{G}(\rho, z) = \frac{8\rho \sin \phi}{\sqrt{\rho^2 + z^2}} \left[ \frac{z}{\rho} \mathbf{a}_\rho - \mathbf{a}_z \right] \]

1.29. Express the unit vector \( \mathbf{a}_x \) in spherical components at the point:

a) \( r = 2, \theta = 1 \text{ rad}, \phi = 0.8 \text{ rad} \): Use

\[ \mathbf{a}_x = (\mathbf{a}_x \cdot \mathbf{a}_r) \mathbf{a}_r + (\mathbf{a}_x \cdot \mathbf{a}_\theta) \mathbf{a}_\theta + (\mathbf{a}_x \cdot \mathbf{a}_\phi) \mathbf{a}_\phi = \sin(1) \cos(0.8) \mathbf{a}_r + \cos(1) \cos(0.8) \mathbf{a}_\theta + (-\sin(0.8)) \mathbf{a}_\phi = 0.59 \mathbf{a}_r + 0.38 \mathbf{a}_\theta - 0.72 \mathbf{a}_\phi \]

b) \( x = 3, y = 2, z = -1 \): First, transform the point to spherical coordinates. Have \( r = \sqrt{14} \), \( \theta = \cos^{-1}(-1/\sqrt{14}) = 105.5^\circ \), and \( \phi = \tan^{-1}(2/3) = 33.7^\circ \). Then

\[ \mathbf{a}_x = \sin(105.5^\circ) \cos(33.7^\circ) \mathbf{a}_r + \cos(105.5^\circ) \cos(33.7^\circ) \mathbf{a}_\theta + (-\sin(33.7^\circ)) \mathbf{a}_\phi = 0.80 \mathbf{a}_r - 0.22 \mathbf{a}_\theta - 0.55 \mathbf{a}_\phi \]

c) \( \rho = 2.5, \phi = 0.7 \text{ rad}, z = 1.5 \): Again, convert the point to spherical coordinates. \( r = \sqrt{\rho^2 + z^2} = \sqrt{8.5}, \ \theta = \cos^{-1}(z/r) = \cos^{-1}(1.5/\sqrt{8.5}) = 59^\circ \), and \( \phi = 0.7 \text{ rad} = 40.1^\circ \). Now

\[ \mathbf{a}_x = \sin(59^\circ) \cos(40.1^\circ) \mathbf{a}_r + \cos(59^\circ) \cos(40.1^\circ) \mathbf{a}_\theta + (-\sin(40.1^\circ)) \mathbf{a}_\phi = 0.66 \mathbf{a}_r + 0.39 \mathbf{a}_\theta - 0.64 \mathbf{a}_\phi \]

1.30. At point \( B(5, 120^\circ, 75^\circ) \) a vector field has the value \( \mathbf{A} = -12 \mathbf{a}_r - 5 \mathbf{a}_\theta + 15 \mathbf{a}_\phi \). Find the vector component of \( \mathbf{A} \) that is:

a) normal to the surface \( r = 5 \): This will just be the radial component, or \(-12 \mathbf{a}_r\).

b) tangent to the surface \( r = 5 \): This will be the remaining components of \( \mathbf{A} \) that are not normal, or \(-5 \mathbf{a}_\theta + 15 \mathbf{a}_\phi\).

c) tangent to the cone \( \theta = 120^\circ \): The unit vector normal to the cone is \( \mathbf{a}_\theta \), so the remaining components are tangent: \(-12 \mathbf{a}_r + 15 \mathbf{a}_\phi\).

d) Find a unit vector that is perpendicular to \( \mathbf{A} \) and tangent to the cone \( \theta = 120^\circ \): Call this vector \( \mathbf{b} = b_r \mathbf{a}_r + b_\phi \mathbf{a}_\phi \), where \( b_r^2 + b_\phi^2 = 1 \). We then require that \( \mathbf{A} \cdot \mathbf{b} = 0 = -12b_r + 15b_\phi \), and therefore \( b_\phi = (4/5)b_r \). Now \( b_r^2[1 + (16/25)] = 1 \), so \( b_r = 5/\sqrt{41} \). Then \( b_\phi = 4/\sqrt{41} \). Finally,

\[ \mathbf{b} = (1/\sqrt{41}) (5 \mathbf{a}_r + 4 \mathbf{a}_\phi) \]