Homework 5 Solutions

1. Signal Space
   (a) The space spanned by this set of signal has a dimensionality of 2 because the basis functions consist of two orthonormal signals. (See part (b)).
   (b) The signals can be represented as a linear combination of the following two orthonormal signals.

   \[
   \begin{align*}
   \varphi_1(t) &= \begin{cases} \frac{2}{T} \cos(2\pi f_c t), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases} \\
   \varphi_2(t) &= \begin{cases} \frac{2}{T} \sin(2\pi f_c t), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}
   \end{align*}
   \]

   (c) Use the trigonometric identity

   \[\cos(A + B) = \cos A \cos B - \sin A \sin B\]

   We can determine the expansion coefficients.

   \[
   \begin{align*}
   s_{11} &= \sqrt{E}, & s_{12} &= -\sqrt{E} \\
   s_{21} &= 0, & s_{22} &= -\sqrt{E} \\
   s_{31} &= -\sqrt{E}, & s_{32} &= -\sqrt{E} \\
   s_{41} &= -\sqrt{E}, & s_{42} &= 0
   \end{align*}
   \]

   (d) The signals are plotted in the following signal space diagram.

   ![Signal space diagram](image)
Note that the signal space diagram is not unique. The constellation can be different if different basis functions are chosen.

2. **Problem 7.19 from Proakis and Salehi.**

1) Since \( m_2(t) = -m_1(t) \), the dimensionality of signal space is 2.

2) As the basis of the signal space, we consider the functions

\[
\psi_1(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t \leq T \\
0 & \text{otherwise} 
\end{cases} 
\]

\[
\psi_2(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\
\frac{1}{\sqrt{T}} & \frac{T}{2} \leq t \leq T \\
0 & \text{otherwise} 
\end{cases} 
\]

3) The vector representation of the signals is

\[
m_1 = [\sqrt{T}, 0] \\
m_2 = [0, \sqrt{T}] \\
m_3 = [0, -\sqrt{T}] 
\]

So the signal constellation is as the following

4) If all the signals are equiprobable then the optimum decision region using minimum distance metric becomes as:

5) If the signals are equiprobable and \( r \) is the received signal vector then,
When \( m_1 \) is transmitted then \( r = \sqrt{T + n_1, n_2} \) and therefore, \( P(e | m_1) \) is written as

\[
P(e | m_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_2 + n_1 > -\sqrt{T})
\]

Since \( n_1, n_2 \) are independent zero mean Gaussian random variables with variance \( \frac{N_0}{2} \), then the random variables \( x = n_1 - n_2 \) and \( y = n_1 + n_2 \) are zero mean Gaussian with variance \( N_0 \). Hence,

\[
P(e | m_1) = \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy
\]

\[
P(e | m_1) = Q\left[\sqrt{T} \sqrt{N_0}\right] + Q\left[\sqrt{T} \sqrt{N_0}\right] = 2Q\left[\sqrt{T} \sqrt{N_0}\right]
\]

When \( m_2 \) is transmitted then \( r = [n_1, \sqrt{T + n_2}] \) and therefore,

\[
P(e | m_2) = P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T})
\]

\[
P(e | m_2) = Q\left[\sqrt{T} \sqrt{N_0}\right] + Q\left[\sqrt{2T} \sqrt{N_0}\right]
\]

From the symmetry of the problem, we obtain \( P(e | m_2) = P(e | m_3) \). Since \( Q[.] \) is monotonically decreasing, we have

\[
Q\left[\sqrt{T} \sqrt{N_0}\right] > Q\left[\sqrt{2T} \sqrt{N_0}\right] \Rightarrow P(e | m_1) > P(e | m_2) = P(e | m_3)
\]

Hence, the message \( m_1 \) is more vulnerable to errors.

3. Union Bound

For the signals given, their signal space representations are shown in the following figure using the same basis functions in Problem 1.
It can determined from the plot that
\[ d_{\text{min}} = \sqrt{2E} \]

A good union bound can be found by considering the two closest neighbors of a signal. For example, for the signal \( s_1 \), we need only to consider the union of the probabilities of detecting \( s_3 \) or \( s_7 \), since the probability of detecting \( s_5 \) is already considered in these cases.

\[
P_e(s_1) \leq \Pr(|r - s_1| > |r - s_3| \mid s_1 \text{ was sent}) + \Pr(|r - s_1| > |r - s_7| \mid s_1 \text{ was sent})
\]

Assuming the signals are equiprobable,
\[
P_e = \sum_i p_i P_e(s_i) = P_e(s_i)
\]
\[
\leq 2 \frac{1}{2} \text{erfc} \left( \frac{d_{\text{min}}}{2\sqrt{N_0}} \right) = \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right)
\]
\[
\leq \frac{1}{\sqrt{\pi}} \exp \left( -\frac{E}{2N_0} \right)
\]

In which the complementary error function is defined as
\[
\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2)dz
\]

For large \( u \), it is bounded by
\[
\text{erfc}(u) \leq \frac{1}{\sqrt{\pi u}} \exp(-u^2) \leq \frac{1}{\sqrt{\pi}} \exp(-u^2)
\]

Another possible answer for the union bound is to consider the sum of all error probabilities. In this case, again using as example the signal \( s_1 \), the error probability is:

\[
P_e(s_1) \leq \Pr(|r - s_1| > |r - s_3| \mid s_1 \text{ was sent}) + \Pr(|r - s_1| > |r - s_7| \mid s_1 \text{ was sent})
\]
\[
+ \Pr(|r - s_1| > |r - s_5| \mid s_1 \text{ was sent})
\]

Now taking into account that the distance between constellation points diagonally opposed is \( 2\sqrt{E} \), the total probability of error results in:
\[
P_e = \sum_i p_i P_e(s_i) = P_e(s_i)
\]
\[
\leq 2 \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right) + \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{N_0}} \right)
\]

The last term in this expression for the probability of error should be very small compared to the first term, so the actual bound would be very similar to the bound obtained before by considering only the two closest neighbors.
4. MAP
Since the a priori probabilities for $H_0$ and $H_1$ are the same, we have

$$\Lambda_0 = \frac{P(H_0)}{P(H_1)} = 1$$

The maximum a posterior rule is as follows

$$H_1: \quad \Lambda \geq \Lambda_0$$
$$H_0: \quad \Lambda < \Lambda_0$$

$$\Lambda = \frac{p_1(y)}{p_0(y)}$$

For $|y| > 1$

$$\Lambda = \infty$$

For $|y| \leq 1$

$$\Lambda = \frac{p_1(y)}{p_0(y)} = \sqrt{\frac{2}{\pi \sigma^2}} \exp \left( -\frac{y^2}{2\sigma^2} \right)$$

Define $\gamma = 2\sigma^2 \ln \sqrt{\frac{2}{\pi \sigma^2}}$. The MAP rule becomes

$$H_1: \quad y^2 \leq \gamma$$
$$H_0: \quad y^2 > \gamma$$

In other words, if $\gamma \geq 0$

$$R_i = \{ y : y \leq \sqrt{\gamma} \text{ or } |y| > 1 \}$$
$$R_o = \{ y : \sqrt{\gamma} < |y| \leq 1 \}$$

If $\gamma < 0$

$$R_i = \{ y : |y| > 1 \}$$
$$R_o = \{ y : |y| \leq 1 \}$$