1. \( S_1 \): For linearity,

\[
T[k_1 x_1(t) + k_2 x_2(t)] = \int_0^t (k_1 x_1(\tau) + k_2 x_2(\tau)) d\tau + (k_1 x_1(t) + k_2 x_2(t)) \\
= \left( \int_0^t \tau k_1 x_1(\tau) d\tau + k_1 x_1(t) \right) + \left( \int_0^t \tau k_2 x_2(\tau) d\tau + k_2 x_2(t) \right) \\
= k_1 \left( \int_0^t \tau x_1(\tau) d\tau + x_1(t) \right) + k_2 \left( \int_0^t \tau x_2(\tau) d\tau + x_2(t) \right) \\
= k_1 T[x_1(t)] + k_2 T[x_2(t)]
\]

Therefore, \( S_1 \) is a linear system.

To see whether the system is time-invariant or not, we compute

\[
z(t) = T[x(t - \sigma)] = \int_0^t \tau x(\tau - \sigma) d\tau + x(t - \sigma) = \int_{-\sigma}^{t-\sigma} (\tau' + \sigma) x(\tau') d\tau' + x(t - \sigma)
\]

where the change of variable \( \tau - \sigma \rightarrow \tau' \) is applied. On the other hand,

\[
y(t - \sigma) = \int_0^{t-\sigma} \tau x(\tau) d\tau + x(t - \sigma)
\]

which is not necessarily the same as \( z(t) \). Therefore, \( S_1 \) is a time-varying system.

\( S_2 \): For linearity,

\[
T[k_1 x_1(t) + k_2 x_2(t)] = \int_0^t [k_1 x_1(\tau) + k_2 x_2(\tau)]^2 d\tau
\]

where one can easily verify that it is not the same as \( k_1 T[x_1(t)] + k_2 T[x_2(t)] \). Therefore, \( S_2 \) is a nonlinear system. To see whether the system is time-invariant or not, we compute

\[
z(t) = T[x(t - \sigma)] = \int_0^t [x(\tau - \sigma)]^2 d\tau = \int_{-\sigma}^{t-\sigma} [x(\tau')]^2 d\tau' \\
= \begin{cases} 
\int_0^{t-\sigma} [x(\tau')]^2 d\tau' + \int_{-\sigma}^{t-\sigma} [x(\tau')]^2 d\tau' = \int_{-\sigma}^{t-\sigma} [x(\tau')]^2 d\tau', & \text{when } \sigma \geq 0, \\
\int_{-\sigma}^{t-\sigma} [x(\tau')]^2 d\tau' & \text{when } \sigma < 0,
\end{cases}
\]

where the change of variable \( \tau - \sigma \rightarrow \tau' \) is applied and the assumption that \( x(t) = 0 \), for \( x < 0 \), is used for the case of \( \sigma \geq 0 \).

On the other hand,

\[
y(t - \sigma) = \int_0^{t-\sigma} [x(\tau)]^2 d\tau
\]

Therefore, two possible answers to this problem are acceptable:
1.) If time-invariability of a system is only considered for the case of $\sigma \geq 0$, we have

$$T[x(t - \sigma)] = y(t - \sigma).$$

Hence, the system is time-invariant.

2.) If time-invariability of a system is considered for both the cases of $\sigma \geq 0$ and $\sigma < 0$, we have

$$T[x(t - \sigma)] \neq y(t - \sigma).$$

Hence, the system is time-varying.

2. (a) The given differential equation can be rewritten as

$$\frac{dy(t)}{dt} = \frac{d(tx(t))}{dt}$$

Then integrating both sides from 0 to $t$ results in

$$y(t) - y(0) = tx(t) - 0 \cdot x(0)$$

Applying the given initial condition $y(0) = 1$, we have

$$y(t) = tx(t) + 1$$

(b) For linearity,

$$T[k_1x_1(t) + k_2x_2(t)] = t[k_1x_1(t) + k_2x_2(t)] + 1 = k_1tx_1(t) + k_2tx_2(t) + 1$$

which is not equal to

$$k_1T[x_1(t)] + k_2T[x_2(t)] = k_1[tx_1(t) + 1] + k_2[tx_2(t) + 1] = k_1tx_1(t) + k_2tx_2(t) + k_1 + k_2$$

Therefore, the system is nonlinear.

To see whether or not the system is time-invariant, we have

$$z(t) = T[x(t - \sigma)] = tx(t - \sigma) + 1$$

which is not equal to

$$y(t - \sigma) = (t - \sigma)x(t - \sigma) + 1$$

Therefore, the system is a time-varying system.

Also, we can see that the output $y(t)$ only depends on the present value of the input $x(t)$. So, the system is causal.
3. (a) Multiplying the given differential equation by $e^t$ results in
\[
\frac{d}{dt}(e^t y(t)) = e^t \left[ \frac{dx(t)}{dt} - x(t) \right]
\]
Then integrating both sides from 0 to $t$ results in
\[
e^t y(t) - e^0 y(0) = \int_0^t e^\tau \left[ \frac{dx(\tau)}{d\tau} - x(\tau) \right] d\tau
\]
which gives
\[
y(t) = y(0)e^{-t} + e^{-t} \int_0^t e^\tau \left[ \frac{dx(\tau)}{d\tau} - x(\tau) \right] d\tau
\]
Note that the right-hand side can be further simplified as follows by applying the integration by part:
\[
y(t) = y(0)e^{-t} + e^{-t} \int_0^t e^\tau \left[ \frac{dx(\tau)}{d\tau} - x(\tau) \right] d\tau
\]
\[
= y(0)e^{-t} + e^{-t} \left[ e^\tau x(\tau)|_0^t - \int_0^t e^\tau x(\tau) d\tau \right] - e^{-t} \int_0^t e^\tau x(\tau) d\tau
\]
\[
= y(0)e^{-t} + e^{-t} [e^t x(t) - x(0)] - 2e^{-t} \int_0^t e^\tau x(\tau) d\tau
\]
\[
= x(t) + [y(0) - x(0)] e^{-t} - 2e^{-t} \int_0^t e^\tau x(\tau) d\tau
\]
With $y(0) = 0$ and $x(0) = 0$, we have
\[
y(t) = x(t) - 2e^{-t} \int_0^t e^\tau x(\tau) d\tau, \quad t \geq 0
\]
(b) Substituting $x(t) = t$ for $t \geq 0$,
\[
y(t) = t - 2e^{-t} \int_0^t e^\tau d\tau
\]
\[
= t - 2e^{-t} \left[ e^\tau - 1 \right]|_0^t
\]
\[
= t - 2e^{-t} \left[ te^t - e^t + 1 \right]
\]
\[
= 2 - t - 2e^{-t}
\]
4. (a) Starting with
\[
z(t) = T[x(t - \tau)] = \int_0^\infty e^{-\sigma} x(t - \sigma - \tau) d\sigma
\]
which is equal to
\[
y(t - \tau) = \int_0^\infty e^{-\sigma} x(t - \tau - \sigma) d\sigma
\]
Therefore, this system is time-invariant.
Now we need to divide the range into the following intervals:

\[ t < 0: \]
\[ y(t) = \int_{0}^{\infty} e^{-\sigma \left\lvert t - \sigma \right\rvert} d\sigma = -t \int_{0}^{\infty} e^{-\sigma} d\sigma + \int_{0}^{\infty} \sigma e^{-\sigma} d\sigma = -t + 1 \]

\[ t \geq 0: \]
\[ y(t) = \int_{0}^{t} e^{-\sigma \left\lvert t - \sigma \right\rvert} d\sigma + \int_{t}^{\infty} e^{-\sigma \left\lvert t - \sigma \right\rvert} d\sigma = t + 2e^{-t} - 1 \]

5. In the following, \( U(t) \) is the unit step function.

(a) nonlinear, time-invariant, causal.

To show that the system is nonlinear, let \( x_1(t) = U(t) \), \( x_2(t) = U(t) \) and \( x_3(t) = x_1(t) + x_2(t) = 2U(t) \). Then, \( y_1(t) = [x_1(t)]^3 = U(t) \), \( y_2(t) = [x_2(t)]^3 = U(t) \), and \( y_3(t) = [x_3(t)]^3 = 8U(t) \neq y_1(t) + y_2(t) \).

(b) linear, time-varying, causal.

To show that the system is time-varying, let \( x(t) = U(t) \). Then, \( y(t) = tx(t) = tU(t) \).
Let \( z(t) = x(t-1) = U(t-1) \), and its corresponding output is \( y_z(t) = tz(t) = tU(t-1) \). Obviously, \( y_z(t) \neq y(t-1) \).

Figure 1: System (a) is nonlinear.
Figure 2: System (b) is time-varying.

(c) linear, time-varying, non-causal.

To show that the system is time-varying, let $x(t) = U(t - 4)$. Then, $y(t) = x(t^2) = U(t - 2) + U(-t - 2)$. Let $z(t) = x(t - 5) = U(t - 9)$, and its corresponding output is $y_z(t) = z(t^2) = U(t - 3) + U(-t - 3)$. Obviously, $y_z(t) \neq y(t - 5)$.

To show that the system is non-causal, let $t = 2$, and then $y(2) = x(4)$, which implies the output at $t = 2$ depends on the “future” input at $t = 4$. So, the system is non-causal.
\[ x(t) = U(t-4) \]
\[ y(t) = U(t-2) + U(-t-2) \]

\[ z(t) = U(t-9) \]
\[ yz(t) = U(t-3) + U(-t-3) \]

Figure 3: System (c) is time-varying.

Figure 4: System (c) is non-causal.