The Nature of Conductors.

Until now we have been considering static electric charges. When charges are in motion, there is a current.

\[ I = \frac{dQ}{dt} \]

where the direction of current is determined by motion of the charges.

The incremental current \( \Delta I \) crossing an incremental surface \( \Delta S \) normal to current density \( J \)

\[ \Delta I = J \Delta S \]

or

\[ I = \int J \cdot dS \]

The current density is related to the velocity of the volume charge density at a point.

\[ \Delta Q = \rho_v \Delta V = \rho_v \Delta S \Delta L \]
If \( \rho_v \) has a z component of velocity, in time \( \Delta t \) the element of charge moves a distance \( \Delta z \).

The resultant current is

\[
\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta s \frac{\Delta z}{\Delta t}
\]

\[
dI = \rho_v \Delta s v_z
\]

\[\text{lim } \Delta t \to 0\]

Thus the resultant current density

\[
J_z = \frac{dI}{ds} = \rho_v v_z
\]

or in general

\[
J = \rho_v v
\]

We call this kind of current density arising from flow or motion of charges "convection current density" and the resultant current "convection current".
The Continuity Equation:

The Principle of Conservation of Charge:

Charges can neither be created nor destroyed; although an equal amount of +ve and -ve charges may be simultaneously created by ionization or removed by recombination of electrons with positive ions.

Let's consider current caused by the flow of charges across a closed surface.

By definition

\[ I = \oint \mathbf{A} \cdot d\mathbf{s} \]

since \( S \) is a closed surface

\[ I = \oint \mathbf{A} \cdot d\mathbf{s} \]

In this case the outward flow of +ve charges must be balanced by a decrease of +ve charge within the closed volume.
\[ I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ_i}{dt} = \int_V (\nabla \cdot \mathbf{J}) \, d\mathbf{v} \]

Here we have used the divergence theorem: viz.

\[ \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V (\nabla \cdot \mathbf{D}) \, d\mathbf{v} = \int_V \mathbf{p}_v \, d\mathbf{v} = Q. \]

Thus

\[ \oint_V (\nabla \cdot \mathbf{J}) \, d\mathbf{v} = -\frac{dQ_i}{dt} = -\frac{d}{dt} \int_V \mathbf{p}_v \, d\mathbf{v} \]

\[ (\nabla \cdot \mathbf{J}) = -\frac{d\mathbf{p}_v}{dt} \quad \text{Continuity Equation} \]

Continuity equation states that current density diverging from a small volume is equal to the time rate of charge decrease per unit volume.
Continuity Equation: An Example

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \mathbf{P}}{\partial t} \]

Let's assume that a current density

\[ J = \frac{1}{r} e^{-t} \quad \text{A/m}^2 \]

is directed radially outward and decreases exponentially with time and as \( 1/r \) in space.

At \( t = 1 \), the total current \( I \) passing through two shells \( r_1 = 5 \text{ m} \) and \( r_2 = 6 \text{ m} \) is

\[ I_1 = J_1 S_1 = \frac{1}{5} e^{-1} \left( 4\pi \times 5^2 \right) = 23.1 \text{ amps} \]

\[ I_2 = J_2 S_2 = \frac{1}{6} e^{-1} \left( 4\pi \times 6^2 \right) = 27.7 \text{ A} \]

How can the total current flowing through \( S_2 \) be greater than \( S_1 \)?
Let's use the continuity equation

\[ \nabla \cdot \mathbf{J} = - \frac{\partial \rho_v}{\partial t} \]

\[ \nabla \cdot \mathbf{J} = \nabla \cdot \left( \frac{1}{r} e^{-t} \mathbf{q}_1 \right) = \frac{1}{r^2} \frac{d}{dt} \left( \frac{r^2 \frac{1}{r} e^{-t}}{r} \right) = \frac{1}{r^2} e^{-t} \]

\[ \nabla \cdot \mathbf{J} \]

\[ - \frac{\partial \rho_v}{\partial t} = \frac{1}{r^2} e^{-t} \]

\[ \rho_v = - \int \frac{1}{r^2} e^{-t} \, dt = \frac{1}{r^2} e^{-t} + \text{const} \]

Boundary condition: as \( t \to \infty \), \( \rho_v = 0 \) \( : \text{const} = 0 \)

\[ \rho_v = \frac{1}{r^2} e^{-t} \quad \text{c/m}^3 \]

but \( \mathbf{J} = \rho_v \mathbf{v} = \frac{1}{r} e^{-t} \)

substitute for \( \rho_v \)

\[ \mathbf{v} = \frac{\rho_v e^{-t}}{\frac{1}{r^2} e^{-t}} = r \text{ m/s} \]

Velocity

At \( r_1 = 5 \text{ m} \), \( \mathbf{v}_1 = 5 \text{ m/s} \) whereas at \( r_2 = 6 \text{ m} \), \( \mathbf{v}_2 = 6 \text{ m/s} \).

This acceleration of charge leads to \( I_2 > I_1 \) while conserving the total charge.
Metallic Conductors.

Energy Band Structure at 0 K

At Room Temp, there may be some electrons in the conduction band. Under the influence of an externally applied electric field, these electrons can flow or move.

Conductor at room temperature

numerus closely spaced energy levels in the conduction band
Electron Motion in a Conductor.

The conduction band electrons move under the influence of an electric field.

In free space
\[ F = -eE = m \ddot{a} \quad \text{Newton's Law} \]
\[ \ddot{a} = -\frac{eE}{m} \]

An accelerating electron would eventually approach \( v = c \).

In a crystalline solid, the velocity approaches a finite value known as the drift velocity \( v_d \) because of collisions
\[ v_d = -\mu_e E \approx 10 \text{ cm s}^{-1} \]

\( \mu_e \) is the electron mobility, \( m^2 \text{s}^{-1} \text{V}^{-1} \).

\( (\mu_e)_{Al} = 1.2 \times 10^{-3} \)
\( (\mu_e)_{Cu} = 3.2 \times 10^{-3} \)
But $\mathbf{J} = \rho_v \mathbf{V}$ and $V_d = -\mu_e E$

Thus in a conductor

$$\mathbf{J} = -\rho_v \mu_e \mathbf{E}$$

We define a new quantity called conductivity of a conductor:

$$\sigma = \frac{\mathbf{J}}{\mathbf{E}}$$

$\sigma$ in Siemans/m

\(\sigma_{Al} = 3.82 \times 10^7\)

\(\sigma_{Cu} = 5.8 \times 10^7\)

and resistivity $\frac{1}{\sigma}$.

Since $\mathbf{J} = -\rho_v \mu_e \mathbf{E} = \sigma \mathbf{E}$

$$\sigma = -\rho_v \mu_e$$

Both $\sigma$ and $\mu_e$ are functions of the temperature $\delta(T)$ increases $\approx 0.4\%$ per $\circ K$ for $Al$, $Cu$

$\Rightarrow$ Resistivity decreases
What is the free electron density in Cu?

\[ \rho_v = \frac{8}{\mu_e} = \frac{5.8 \times 10^9}{3.2 \times 10^3} \text{ cm}^{-3} \]

\[ = 1.8 \times 10^{10} \text{ cm}^{-3} \text{ or } 1.8 \times 10^4 \text{ C cm}^{-3} \]

Then \( e n_e = \rho_v \)

\[ n_e = \frac{1.8 \times 10^4}{1.6 \times 10^{-19}} = 1.125 \times 10^{23} \text{ cm}^{-3} \]

This is a huge number.

The interparticle spacing is 0.1125 Å.
Resistance of a Conductor and Ohm's Law:

Let's assume that both $I$ and $E$ are uniform in the cylindrical region as shown.

$I = JS$

$\mathbf{S} \rightarrow \delta$

$E = \frac{V}{L}$

$a \rightarrow L \rightarrow b$

$I = \int_{a}^{b} J \cdot dl = JS$

$V_{ab} = -\int_{b}^{q} E \cdot dl = -E \int_{b}^{q} dl = -E \cdot L_{ba}$

$= E \cdot L_{ab} \text{ or } V = EL$

$J = \frac{I}{S} = \delta E = \delta \frac{V}{L}$

$V = \frac{L}{\delta s} I = R I \quad \text{Ohm's Law}$

where $R = \frac{L}{\delta s}$

Resistance of the device

More Generally, for a non-uniform material

$R = \frac{V_{ab}}{I} = -\frac{\int_{b}^{q} E \cdot dl}{\int_{s}^{a} \delta E \cdot ds}$
Conductor Properties and Boundary Conditions

If free charges are introduced in the body of a conductor, they repel one another and move until they reach and reside on the surface as a surface charge density $\rho_s$. In steady state, no current flows on the surface, i.e., $E_t = 0$.

By Gauss' Law, $\oint \rho_s dS = \oint \mathbf{D}_n dS$

$\therefore \mathbf{D}_n = \rho_s$ or $E_0 E_N = \rho_s$

$E_N = \frac{\rho_s}{E_0}$

Since $E_t = 0$, $\oint E_t \cdot dL = 0$. Conductor surface is an equipotential surface.