Example:

If \( H = 0.2 z^2 q_x \) for \( z > 0 \)

\( H = 0 \) everywhere else

Calculate \( \oint \mathbf{H} \cdot d\mathbf{l} \) about a square path with side \( d \)
centered at \( (0, 0, z_1) \) in the \( y = 0 \) plane where \( z_1 > z_0 \)

\[
\oint \mathbf{H} \cdot d\mathbf{l} = 0.2 \left( z_1 + \frac{1}{2} d \right)^2 d - 0.2 \left( z_1 - \frac{1}{2} d \right)^2 d
\]

\[= 0.4 \pi d^2 \]

In the limit as the area approaches zero

\[
(\nabla \times \mathbf{H})_y = \lim_{d \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{l}}{d^2} = \frac{0.4 \pi d^2}{d^2} = 0.4 \pi
\]

So \( \nabla \times \mathbf{H} = 0.4 \pi z_1 q_y \)
We can get the same answer from the definition of $\nabla \times \mathbf{H}$

$$\nabla \times \mathbf{H} : \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z} \left( 0.2 z^2 \right) a_y = 0.42 a_y$$

which is the same as the above result.

Stoke's Theorem

Consider the surface $S$ which is broken up into incremental surfaces of area $\Delta S$. If we apply the definition of curl to one of these incremental surfaces then,

$$\oint \mathbf{H} \cdot dL_{\Delta S} = \left( \nabla \times \mathbf{H} \right)_{\Delta S}$$

where the $N$ subscript indicates the right-handed normal to the surface.

$dL_{\Delta S}$ indicates that the path $dL$ is the perimeter of $\Delta S$.

$$\oint \mathbf{H} \cdot dL_{\Delta S} = \left( \nabla \times \mathbf{H} \right) \cdot a_N$$
If we determine this circulation for every $\Delta S$ comprising surface area $S$ and sum the result, some cancellation will occur because every interior wall is covered once in each direction.

contribution adds on the boundary

cancellation on the inside.

However, there is an cancellation on the boundaries. Thus:

$$\oint_{\partial \Sigma} \mathbf{H} \cdot d\mathbf{l} = \int_{\Sigma} (\nabla \times \mathbf{H}) \cdot d\mathbf{s}$$

where $\partial \Sigma$ bounds the surface area $\Sigma$.

This is known as Stoke's Theorem.

Now we can obtain Ampere's Circuital Law from Stoke's Theorem:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \text{and} \quad \oint_{\partial \Sigma} \mathbf{H} \cdot d\mathbf{l} = \int_{\Sigma} (\nabla \times \mathbf{H}) \cdot d\mathbf{s}$$

$$\oint_{\partial \Sigma} \mathbf{J} \cdot d\mathbf{l} = \int_{\Sigma} (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = I = \oint_{\partial \Sigma} \mathbf{H} \cdot d\mathbf{l} \quad \checkmark$$
Magnetic Flux and Magnetic Flux Density.

In free space, magnetic flux density $B$ is defined as

$$ B = \mu_0 H $$

Webers per m$^2$ or Tesla

$\mu_0$ permeability of free space

$\mu_0 = 4\pi \times 10^{-7}$ H/m

Other units for $B$ are Gauss

1 Tesla $= 10^4$ Gauss

Then the magnetic flux $\Phi$

$$ \Phi = \oint_S B \cdot ds \text{ Wb} $$

Note analogy with electrostatics

$$ \mathbf{D} = \varepsilon \mathbf{E} $$

$$ \psi = \oint_S \mathbf{D} \cdot ds = Q. $$

But since there are no magnetic charges, that are the origin of magnetic field intensity

$$ \oint_S B \cdot ds = 0 = \int_V (\nabla \times B) \, dv $$

$$ \therefore \nabla \cdot B = 0 $$
Equations of Electrostatics and Magnetostatics in Point Form

\[ \nabla \cdot D = \rho_v \]
\[ \nabla \times E = 0 \]
\[ \nabla \times H = J \]
\[ \nabla \cdot B = 0 \]
\[ E = -\nabla V \]

Equations of Electrostatics and Magnetostatics in Integral Form.

\[ \oint_S D \cdot ds = \int_V \rho_v dv = Q \]
\[ \Phi_E \cdot dl = 0 \]
\[ \oint_S H \cdot dl = I = \int_C I \cdot ds \]
\[ \oint_S B \cdot ds = 0 \]
The Scalar and Vector Magnetic Potential

\[ \mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{E} = -\nabla \mathbf{V} \]

then if

\[ \mathbf{B} = \mu \mathbf{H} \]

is there a scalar magnetic potential \( V_m \)

\[ \mathbf{H} = -\nabla V_m \]

but any definition must be consistent with other relationships:

\[ \nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m) \]

but \( \nabla \times \nabla \) of a scalar = 0

\[ \therefore \mathbf{H} = -\nabla V_m \] is only true if \( \mathbf{J} = 0 \)

In many magnetic problems, current carrying wires occupy only a small fraction of the volume. Excluding this, the concept of a scalar magnetic potential is very useful. In this "source free" region the scalar potential satisfies Laplace's equation

\[ \nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0 \]

\[ \mu_0 \nabla \cdot (-\nabla V_m) = 0 \]

\[ \therefore \nabla^2 V_m = 0 \quad (\mathbf{J} = 0) \]
Limitations of the Magnetic Scalar Potential.

\[ E = -\nabla V \]

The scalar electric potential is a single valued function of position. Once a zero reference is assigned there is only one value of \( V \) associated with each point in space. What about \( V_m \)?

Consider the cross-section of the co-axial line as shown.

In the region \( a < \rho < b \), \( \mathbf{J} = \mathbf{0} \) and we specify \( V_m \).

However \( \mathbf{H} = \frac{I}{2\pi\rho} \quad \phi \)

\[ \therefore \frac{I}{2\pi\rho} = -\nabla V_m \bigg|_{\phi} = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi} \]

\[ \therefore \frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi} \]

or \( V_m = -\frac{I}{2\pi} \phi \)

Now you can see that for \( \phi = n(2\pi) \) we return to the same point in space but \( V_m = -n\left(\frac{I}{2\pi}\right) \) ie multiple valued.
The Magnetic Vector Potential.

\[ \mathbf{B} = \nabla \times \mathbf{A} \quad \text{a magnetic vector potential}. \]

This definition must be consistent with

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{H} = \mathbf{J}, \quad \mathbf{B} = \mu \mathbf{H} \]

Now

\[ \nabla \cdot \nabla \times \mathbf{A} = 0 \quad \text{divergence of curl of a vector field is always } 0 \]

and \[ \nabla \times \mathbf{H} = \frac{1}{\mu} \nabla \times \nabla \times \mathbf{A} = \mathbf{J} \text{ need not be zero} \]