We saw that in an inhomogeneously broadened laser multimode oscillation can take place due to "hole-burning".

What happens if the phases of all these modes are "locked" at some point in time?

The total electric field

$$e_{\text{total}}(t) = \sum_{n=1}^{N} E_n(t) \exp \left[ -j \frac{(n-1)}{2} \right] \left( \omega_n \Delta \nu_c \right) t + \phi_n(t)$$

$$= \left( \frac{N-1}{2} \right)$$

$$w_c = 2\pi f_c = 2\pi \left( \frac{c}{2d} \right)$$

Let $$\phi_n(t) = \text{const} = \theta$$

and $$E_n = E_0$$ equal amplitude modes

$$e_{\text{total}}(t) = \sum_{n=1}^{N} e^{j \omega_n t} e^{j \Delta \nu_c t} = e^{j \omega_0 t} e^{j \Delta \nu_c t} \sum_{n=1}^{N}$$

$$= e^{j \omega_0 t} e^{j \Delta \nu_c t} \sum_{n=1}^{N}$$
Let $k = \frac{(N-1)}{2}$.

\[
\frac{e^{j\omega c t (N-1)}}{e^y} = e^{\sum x^n} - x^{-k}
\]

Consider the sum

\[
S = x + x + \ldots + x + x
\]

\[
xS = x^{-k+1} + x^k + x^{k+1}
\]

\[
(1-x)S = x - x
\]

\[
S = x + x^{k+1}
\]

\[
\frac{x - x^{-k}}{1-x} = \frac{x^{k+1} - x^{-k}}{(x-1)}
\]

\[
x^{1/2} - x^{-1/2}
\]

\[
jN\omega ct/2 - jN\omega ct/2
\]

\[
\frac{e^{j\omega ct/2}}{e^{j\omega ct/2}}
\]

\[
\frac{\sin \left( \frac{N\omega ct}{2} \right)}{\sin \left( \frac{\omega ct}{2} \right)}
\]

where $k = \frac{N-1}{2}$.
\[ E_{\text{total}}(t) = E_0 e^{j\omega_0 t} \sin \left( N \frac{\omega_0 t}{2} \right) \]

\[ I(t) = \frac{e(t)e^*(t)}{2\eta_0} = \frac{E_0^2}{2\eta_0} \left[ \frac{\sin \left( N \frac{\omega_0 t}{2} \right)}{\sin \left( \frac{\omega_0 t}{2} \right)} \right]^2 \]

Each time the numerator goes through N cycles each time the denominator goes through one cycle.

Let's look at the denominator:

\[ \frac{\omega_0 t}{2} = \frac{\pi}{2} \text{ max} \]
\[ \frac{\omega_0 t}{2} = \pi \text{ min} \]
\[ \omega_0 t = 2\pi \left( \frac{c}{2d} \right) \]
\[ \frac{2\pi}{c} \frac{t}{c} = \frac{\pi}{2} \ \text{Round trip} \]

where the round trip is \( 2d/c \).

Inhomogeneously broadened gain medium:

Ball of photons as a result of interference of longitudinal modes.
\[ P_{\text{peak}} = N P_{\text{ave}} \]

But \( N = \frac{\Delta \nu_{\text{Doppler}}}{c/2d} \)

Pulse length \( \sim \frac{\text{Round trip transit time}}{N} \)

Example:

Consider Nd: glass laser

\[ \Delta \lambda = 10^{-6} \text{A} \]

\[ \Delta \nu = \frac{c}{\lambda^2} \Delta \lambda = 3 \times 10^{12} \text{Hz} \]

Longitudinal modes \( N = \frac{\Delta \nu}{\text{FSR}} \) for a 1m long cavity

\[ N = \frac{3 \times 10^{12} \times 2}{3 \times 10^8} = 2 \times 10^4 \]

\[ T_{\text{pulsed}} \sim \frac{1}{\Delta \nu} \sim 300 \text{fs} \]

This pulse occupies only

\[ L = \frac{cT_{\text{pulsed}}}{2} = \frac{3 \times 10^8 \times 0.3 \times 10^{-12}}{2} = 90 \text{mm of space} \]

Explain how active and passive mode-locked lasers work.