Gain Saturation in a Homogeneous Broadened Transition

Mathematical Description.

For a homogeneous broadened transition

\[
\gamma(v) = \gamma_0(v_0) \left[ \frac{(\Delta v/2)^2}{(v-v_0)^2 + (\Delta v/2)^2} \right]^{1.4}
\]

Let's assume that \( \Delta v = 1 \text{GHz} \) and that the small signal gain coefficient at the line center \( v_0 \) is \( N \) times the loss coefficient \( \alpha \)

\[
\frac{\gamma_0(v_0)}{\alpha} = N
\]

\[
\gamma_0(v) = \alpha = \gamma_0(v_0) \left[ \frac{1}{(v-v_0)^2 + (\Delta v/2)^2} \right], \quad \text{or} \quad \frac{\alpha}{\gamma_0(v_0)} = \frac{1}{N}
\]

Clearly, the gain exceeds the losses over the frequency interval

\[
2|v-v_0| \leq (N-1)^{1/2} \Delta v
\]

For instance \( \gamma_0(v_0) = 4\alpha \) and \( \Delta v = 1 \text{GHz} \), there is a band \( 1.7 \text{GHz} \) wide where laser oscillation can take place. No let's assume we restrict laser oscillation to TEM\(_{00} \) mode. The longitudinal modes are \( c/2d \) apart. For a 1 meter long cavity the mode separation is \( 1.5 \times 10^8 \text{Hz} \). Thus at least to longitudinal modes can be excited.
Now to describe the final laser state we need a mathematical description of gain saturation.

![Diagram of laser system](image)

\( R_1 \) : direct excitation of 1 from G.S. + any indirect routes such as excitation to a higher state followed by spontaneous emission from a higher state to state 1.

\( R_2 \) : similar to \( R_1 \).

We assume \( N_1 + N_2 \ll N_0 \) thus \( N_0 \) can be thought as independent of pumping rates. The decay rates are specified as the inverse of a lifetime for a certain process to take place.
1) The spontaneous emission rate and any other process that decreases $N_2$ and simultaneously increases $N_1$ is denoted by $1/\tau_{21}$.

2) The rate of deactivation of state (2) that bypasses (1) is denoted by $1/\tau_{20}$.

3) The lifetime of (2) is $\tau_2^{-1} = \tau_{21}^{-1} + \tau_{20}^{-1}$

4) The decay rate of state 1 due to any and all causes is denoted by $1/\tau_1$.

The simulated emission and reabsorption rates are given by $B_{21} g(v) \Phi_\nu$ and $B_{12} g(v) \Phi_\nu$, respectively. Thus, the rate equations for $N_2$ and $N_1$ are

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - \left[ B_{21} g(v) \Phi_\nu \right] N_2 + \left[ B_{12} g(v) \Phi_\nu \right] N_1$$

$$\frac{dN_1}{dt} = R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{\tau_2} + \left[ B_{21} g(v) \Phi_\nu \right] N_2 - \left[ B_{12} g(v) \Phi_\nu \right] N_1$$

Let's assume that the degeneracies of the states $g_1$ and $g_2$ are equal. Furthermore:

$$\frac{A_{21}}{A_{21}} = \frac{8 \pi m \nu^3}{c^4}$$

$$B_{21} g(v) = \frac{c}{\hbar \nu} \lambda^2 A_{21} g(v) = \frac{c}{\hbar \nu} \delta(v)$$

with $\Phi_\nu = \nu I_v / c$.
At equilibrium \( \frac{d}{dt} = 0 \) and we obtain a set of linear algebraic equations

\[
R_2 = \left[ \frac{1}{\tau_2} + \frac{\sigma(v) I_v}{h\nu} \right] N_2 - \frac{\sigma(v) I_v}{h\nu} N_1
\]

\[
R_1 = - \left[ \frac{1}{\tau_2} + \frac{\sigma(v) I_v}{h\nu} \right] N_2 + \left[ \frac{1}{\tau_1} + \frac{\sigma(v) I_v}{h\nu} \right] N_1
\]

We want an expression for the population inversion

\[
N_2 - N_1
\]

\[
N_2 - N_1 = \left\{ \frac{R_1 \tau_2}{1 + \left( \tau_1 + \frac{\tau_2}{\tau_2 - \tau_1} \frac{1}{\tau_2} \right) \left[ \frac{\sigma(v) I_v}{h\nu} \right]} \cdot \left( 1 - \frac{\tau_1}{\tau_2} \right) - R_1 \tau_1 \right\}
\]

but \( \sigma(v)(N_2 - N_1) = \gamma(v) \) The gain coefficient

\[
\gamma(v) = \left\{ \sigma(v) \right\}
\]

Now let's examine the behaviour of \( \gamma(v) \) for small intensities.

\[
\gamma(v) = \gamma_0(v) = \left[ \frac{R_2 \tau_2}{1 - \frac{\tau_1}{\tau_2}} - R_1 \tau_1 \right] \sigma(v)
\]

\[
\gamma_0(v) = \frac{1}{1}
\]

I\_v small
Hence  
\[ \delta(v, I_v) = \frac{\delta_0(v)}{1 + \left( \frac{\tau_1 + \tau_2 - \tau_1 \tau_0 / \tau_2}{\tau_2} \right) \left[ d(v) I_v / I_s \right]} \]

\[ = \frac{\delta_0(v)}{1 + \left[ I_v / I_s \right] \bar{g}(v)} \]

where  \( I_s \) is the saturation intensity  \( \delta(v) = \delta(v_0) \bar{g}(v) \).

\[ I_s = \frac{hv}{\tau_1 + \tau_2 - \tau_1 \tau_0 / \tau_2} \frac{1}{d(v_0)} \]

and \( \bar{g}(v) \) is the line shape function normalized to be unity at line center. For a Lorentzian

\[ \bar{g}(v) = \frac{(\Delta v/2)^2}{(v - v_0)^2 + (\Delta v/2)^2} \]

For many lasers \( \tau_1 \ll \tau_2 \) and the lifetime of state 2 is approximately dictated by radiation.

\[ I_s = \frac{hv}{\tau_2 d(v_0)} \]

Equation 11.12 gives us the general requirements for achieving a population inversion and gain. To get a large gain we want the pumping rate of the upper state \( R_2 \) as large as possible, keeping \( R_1 \) as small as possible.
Lifetime of state 2 should be long whereas that of state 1 should be short.

\[ \tau_2 = \tau_{21} + \tau_{20} \]

The limiting lifetime is \( \tau_{21} \). Clearly if \( \tau_{21} \rightarrow \infty \) then \( A_{21} \rightarrow 0 \) and the cross-section \( \sigma(v) \rightarrow 0 \).

Lifetime of state 1 should be as short as possible. In other words, the population \( N_1 \) should be depleted as fast as possible. Indeed, the depletion of the lower state and pumping of the upper state are the rate-limiting processes for a good laser.

\[ \chi(v) = \frac{\chi_0(v)}{1 + (I(x)/I_s) \bar{g}(v)} \]

This saturation law indicates that the gain coefficient depends on the intensity of radiation being amplified. Thus, the rate at which the intensity increases with distance is given by

\[ \frac{dI(x)}{dz} = \frac{\chi_0(v) I(x)}{1 + (I(x)/I_s) \bar{g}(v)} = \chi(v, I) I(x) \]

In a laser amplifier of length \( d \), the output intensity
\[ I_v(z=d) \]
\[ \int dI_v \left[ \frac{1}{I_v} + \frac{\bar{g}(v)}{I_s} \right] = \int \gamma_o(v) \, d\bar{z} \]
\[ I_v(z=0) \]

\[ \ln \left( \frac{I_v(d)}{I_v(0)} \right) = \frac{\bar{g}(v)}{I_s} \left[ I_v(d) - I_v(0) \right] = \gamma_o(v) \, d \]

This is a transcendental equation that must be solved numerically for the output in terms of the input.

Limiting Cases:

1. If \( I_v(d) \) and \( I_v(0) \) is very much smaller compared to \( I_s \) then we obtain the small-signal gain

\[ \ln \left( \frac{I_v(d)}{I_v(0)} \right) = \gamma_o(v) \, d \]

\[ I_v(d) = I_v(0) \exp (\gamma_o(v) \, d) \]

2. If \( I_v(0) \gg I_s \)

\[ I_v(d) = I_v(0) + \frac{\gamma_o(v) I_s \, d}{\bar{g}(v)} \]

Implicit in the derivation is the assumption that all atoms have the same line shape \( g(v) \). Thus if one atom in level 2 gave up its energy to the stimulating field, the gain coefficient
Physical Description of Laser Oscillation in an Inhomogeneous Broadened Transition.

Consider an inhomegeneous broadened line shape such as from atoms arranged in a crystalline structure. Because of their locations at different sites in a crystal lattice, the center frequencies of the groups are different. In the laboratory we measure a convolution of line shapes from each of these groups.

![Diagram of line shapes and stimulating wave](image)

Hypothetical inhomogeneous line shape.

Now suppose we have a stimulating wave at frequency \( \nu \). It can interact well with group 4 atoms but they comprise only 20% of the total atoms. It can also interact with atoms of group 3 (40% of the population) with 25% of the peak interaction strength. There is very little interaction with other groups. Thus we can deplete the population inversion in group 4 easily group 3 reasonably but not groups 1, 2, or 5.
Evolution of Laser Oscillation in a Doppler Broadened Transition:

Hole Burning and Lamb Dip.

The initial phase of the build-up of radiation in the laser cavity is identical for inhomogeneous and homogeneous transitions. The spontaneous emission goes into cavity modes more-or-less like the line shape and each cavity mode grows according to a gain-loss formula. The difference becomes startling near saturation. The field interacts with a specific group of atoms according to the homogeneous line shape in the atoms' frame of reference.

Consider a Doppler broadened transition with center frequency $\nu_0$. The condition for the maximum strength of interaction (emission or absorption) between the moving atoms and the wave is that the Doppler shifted frequency $\nu_D$ seen by the atom be equal to the atomic resonant frequency.

$$\nu_D = \nu \left(1 - \frac{v_2}{c}\right) = \nu_0$$

where $\nu$ is the frequency of the e.m. wave in the lab frame and $v_2$ is the velocity of the atoms.
Or we might say that a wave of frequency $v$ moving through a medium will "seek-out" and interact most strongly with those atoms whose velocity component $v_2$ satisfies

$$v = \frac{v_0}{1 - v_2/c} \cdot v_0 \left(1 + \frac{v_2}{c}\right)$$

for $v_2 \ll c$.

Now consider a gas laser oscillating on a single longitudinal mode with frequency $v > v_0$. The standing wave e.m. field inside the laser resonator consists of two wave travelling in opposite directions. Since $v > v_0$ the wave will interact with atoms having $v_2 > 0$ i.e.

$$v_2 = + \frac{c}{v} (v - v_0)$$

The wave travelling in the opposite $-\varepsilon$ direction interact with atoms with

$$v_\varepsilon = - \frac{c}{v} (v - v_0)$$

So that once again $v_0 = v_0$. Thus, due to standing wave nature of the field inside a conventional two-mirror laser oscillator, a given frequency of oscillation interacts with two velocity classes of atoms.
If the velocity distribution function of atoms in the upper laser level is given by

\[ f(V_z) = e^{-\frac{MV_z^2}{2kT}} \]

then as these atoms undergo stimulated emission, the velocity distribution function develops two depressions at

\[ |V_z| = \frac{c}{\nu} (\nu - \nu_0) \]

If the oscillation frequency \( \nu = \nu_0 \) only a single hole exists in the velocity distribution function of the inverted atoms, centered on \( V_z = 0 \). Thus we may expect the power output of the laser oscillating at \( \nu = \nu_0 \) to be less than that of a laser in which \( \nu \) is slightly less or greater than \( \nu_0 \). In a laser oscillator this can be achieved by moving one of the laser mirrors. This 'dip' in the output power is known as 'lamb dip'.
The output power as a function of frequency of a single mode 1.15 μm He-Ne laser using Ne²⁺ isotope.

**Multimode Oscillation** (Spatial Hole Burning)

When a laser is operated in a multimode configuration situation becomes quite complicated. Not only are the resonant frequencies of TEM\(_{m,n,n}\) modes different (i.e., they compete for different atoms) but also they may occupy different regions in space. Because of this latter characteristic even a homogeneously broadened transition can support multimode oscillations by spatial hole burning.