Basics of the Gain Medium

Laser Cavity

As we saw, the laser cavity determines the transverse field distribution (modes) and frequencies (longitudinal modes). However, it is the gain medium that provides the photons at these characteristic resonant frequencies.

For simplicity, we treat the gain medium as a two-level atomic system with energy levels 1 and 2 having energies $E_1$ and $E_2$. 

\[ E_2 - E_1 = h\nu \]
Before

\[ p(\nu) \]

\[ B_{21} p(\nu) \]

Spontaneous Emission or Radiative Decay

Stimulated Emission

\[ h\nu = E_2 - E_1, \text{ random direction, phase, polarization} \]

Absorption

\[ \text{Photons in a narrow frequency band around } E_2 - E_1 \text{ are absorbed. Rate depends on intensity of photons near } h\nu \]

Stimulated Emission

\[ \text{Same frequency, phase, direction, polarization as incoming photons} \]
\[
\frac{dN_2}{dt} = -A_{21}N_2 + B_{12}p(v)N_1 - B_{21}p(v)N_2 = -\frac{dN_1}{dt}
\]

Spontaneous Absorption Stimulated
emission Emission

At steady state, \( \frac{d}{dt} = 0 \), which occurs at thermal equilibrium

\[
\frac{N_2}{N_1} = \frac{B_{12}p(v)}{A_{21} + B_{21}p(v)}
\]

But at equilibrium \( \frac{N_2}{N_1} \) can be predicted by the Maxwell-Boltzmann relationship

\[
\frac{N_2}{N_1} \propto \frac{g_2}{g_1} \exp \left(-\frac{\Delta E}{kT}\right) = \frac{g_2}{g_1} \exp -\frac{hv}{kT}
\]

in general

\[
\frac{N_i}{N_j} \propto \frac{g_i}{g_j} \exp -\frac{hv_{ij}}{kT}
\]
\[ \frac{g_2}{g_1} \exp - \frac{hv}{kT} = \frac{B_{12} \rho(v)}{A_{21} + B_{21} \rho(v)} \]

\[ A_{21} \left( \frac{g_2}{g_1} \exp - \frac{hv}{kT} \right) + B_{21} \rho(v) \left( \frac{g_2}{g_1} \exp - \frac{hv}{kT} \right) = B_{12} \rho(v) \]

For now accept \( g_2/g_1 = (2J_z+1)/(2J_1+1) \) ratio of degeneracies of the energy levels.

Collecting terms in \( \rho(v) \)

\[ \rho(v) = \frac{A_{21} \left( \frac{g_2}{g_1} \exp - \frac{hv}{kT} \right)}{B_{12} - B_{21} \left( \frac{g_2}{g_1} \exp - \frac{hv}{kT} \right)} \]

divide by \( \frac{g_2}{g_1} \exp - \frac{hv}{kT} \)

\[ \rho(v) = \frac{A_{21}}{B_{21}} \left( \frac{1}{B_{12} g_2 \exp + \frac{hv}{kT} - 1} \right) \]

but at thermal equilibrium

\[ \rho(v) = \frac{8 \pi h v^3}{c^3} \frac{1}{\exp \frac{hv}{kT} - 1} \]

Planck's Radiation Law
The two expressions must be the same since both are valid at thermal equilibrium.

Thus, by inspection

\[
\frac{A_{21}}{B_{21}} = \frac{8\pi h v^3}{c^3}
\]

and

\[
\frac{B_{12}}{B_{21}} \frac{g_2}{g_1} = 1
\]

or

\[
\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}
\]

These relationships between the \(A_{21}\), \(B_{12}\) and \(B_{21}\) coefficients are called Einstein relationships.
Decay of Excited Atoms

Real atoms have numerous energy levels for excited electrons. Once in an excited state, an electron can decay radiatively (spontaneously) or because of inelastic collision with a free electron.

Here is an atom with 3 energy levels

\[ E_0 \quad \text{Ground} \]
\[ E_1 \]
\[ E_2 \]

\[ k_{10} \]
\[ A_{20} \]
\[ A_{11} \]
\[ k_{20} \]
\[ k_{11} \]

Let \( k_{ij} \) be the collisional de-excitation rates.

\[
\frac{dN_2}{dt} = - \left( A_{20} + A_{11} \right) N_2 - \left( k_{20} + k_{11} \right) N_2
\]

\[
= - \left( A_2 + k_2 \right) N_2 \quad \text{where} \quad A_2 = A_{20} + A_{21}
\]

\[
= - k_2 N_2 \quad \text{total decay rate}
\]
\[ N_2(t) = N_2(0) \exp(-\gamma_2 t) \]

\[ \gamma_2 = \frac{1}{\tau_2} \quad \text{where } \tau_2 \text{ is the lifetime of level } 2 \]

\[ \tau_2 = \frac{1}{A_2 + k_2} \quad \text{Fluorescence lifetime of level } 2 \]

If we could eliminate all the collisional processes \( k_2 = 0 \), then

\[ \tau_2 = \tau_r = \frac{1}{\sum_{i \neq j} A_{ij}} \quad \text{Radiative lifetime of level } 2 \]

Branching ratio \( \phi_{2 \rightarrow 1} \) is the fraction of radiative decay of level 2 that ends up in level 1

\[ \phi_{2 \rightarrow 1} = \frac{A_{21}}{\sum_{n \neq 2} A_{2n}} \]

Each of these \( 2 \rightarrow 1 \) transitions contributes a photon of energy \( h\nu_{21} \) over \( 4\pi \) steradian. Therefore the power emitted in angle \( d\Omega \) is

\[ dP = \frac{d\Phi}{4\pi} A_{21} N_2(t) \ h\nu_{21} \]
and substituting for $N_2(t)$

$$dP(t) = \frac{dN}{4\pi} A_{21} N_2(0) e^{-\gamma_2 t}$$

From communications we know that if a signal has an envelope (in this case: an exponential decay) there will be a bandwidth associated with it.

Let's define an electric field that has a carrier frequency $\omega_0$ that leads to an exponentially decaying power as shown above: $-\frac{\gamma_2}{2} t$

$$E(t) = E_0 e^{-\gamma_2 t} \cos \omega_0 t$$

where $\gamma_2$ is the field decay coefficient whereas $\gamma_2$ is the power decay rate.
The frequency spectrum $E(w)$ of the signal can be found by

$$E(w) = \mathcal{F} \left\{ E(t) \right\} = \int_0^\infty E(t) e^{-j\omega t} \, dt.$$ 

$$E(w) = \frac{\gamma l_2 + j\omega}{w_0^2 + w^2 + (\gamma l_2)^2 + j\omega l_2}$$

$\delta_1$ replaced by $\gamma$

To find the spectral distribution of power we need

$$E(w) E^*(w) = S(w) \quad (\text{Also called Spectral Power Density})$$

$$S(w) = \frac{(\gamma l_2)^2 + w^2}{E_0 \left[ (w_0^2 - w^2) + (\gamma l_2)^2 \right]^2 + (\delta l)^2}$$

Now we make some approximations. Let's assume that $\gamma \ll \omega$ which means that the frequency spectrum is going to be strongly peaked around $w_0$. So we set $w_0 + \omega = 2w_0$, but keep the difference explicitly.

Also $(\gamma l_2)^2$ small cf. $w^2 \approx w_0^2$
\[ S(w) = \frac{w_0^2}{\varepsilon_0^2} \frac{1}{(w_0 + w)^2 (w_0 - w)^2 + (\omega L)^2} \]

\[ = \frac{1}{4} \frac{1}{(w_0 - w)^2 + \left( \frac{\omega L}{2} \right)^2} \]

This type of functional dependence is called a Lorentzian.

Suppose we want to turn this into a probability function by with the normalization

\[ \int_{-\infty}^{\infty} g(v) dv = 1 \]

Then we must write

\[ \frac{S(v)}{\varepsilon_0^2} = \frac{1}{4} \frac{1}{(2\pi)^2} \frac{1}{(v_0 - v)^2 + \left( \frac{\Delta v}{2} \right)^2} \]

Write \( \frac{\omega L}{2\pi} = \Delta v \)

\[ = \frac{1}{4} \frac{1}{(2\pi)^2} \frac{k}{(v_0 - v)^2 + \left( \frac{\Delta v}{2} \right)^2} = g(v) \]

and find a constant \( k' \) that gives \( \int g(v) dv = 1 \)
\[ \int_0^\infty g(v) dv = \int_0^\infty \frac{k' dv}{(v-v')^2 + (\Delta v/2)^2} = 1 \]

which leads to

\[ k' = \Delta v/2\pi \]

and

\[ g(v) = \frac{\Delta v}{2\pi \left[ (v_0-v)^2 + (\Delta v/2)^2 \right]} \]

We call \( g(v) \) the line shape function. It is strongly peaked about \( v = v_0 \) and has a full width of \( \Delta v = \frac{\delta}{2\pi} \).
but $y = y_2$

$$\Delta \nu = \frac{1}{2\pi} \left[ A_2 + k_2 \right]$$

$$\Delta \nu_{\text{min}} = \frac{1}{2\pi} \left[ A_{21} \right]$$

Now we have a physical meaning to the bandwidth of the signal. It is due to the finite lifetime of the excited level. Now we can think about the photons being emitted because of a transition between two levels that are not infinitely sharp but blurred out with a lineshape such that photons with energies greater and less than $\nu_{12}$ can be emitted.

\[ \begin{array}{c}
\text{Lorentzian Profile}.\\
E_2 \quad h\nu_{21} \\
\text{in the picture above } E_1 \text{ is still shown to be infinitely sharp because it is assumed to be the ground state.}
\end{array} \]

IE Once an electron is in this state it can no longer radiatively decay. The state has an $\infty$ lifetime.
This type of line broadening is the same for every atom. It comes about because the excited states have a finite lifetime. If the did not, they would not decay and there would be no photon emission.

The broadening of the spectrum arising from this is called "lifetime broadening". It is the minimum bandwidth that any optical signal must have. Because lifetime broadening is identical for all atoms it is called homogeneous broadening: \( g_h(\nu) \).

Now we can give a physical meaning to \( g(\nu) \) the lineshape function:

\[
g(\nu) d\nu
\]

1. It is the probability that when an excited state electron decays by radiative decay that it will emit a photon with energy between \( \nu \) and \( \nu + d\nu \).

2. It is also the probability that a photon between the frequencies \( \nu \) and \( \nu + d\nu \) will be absorbed by the atom or will cause stimulated emission from an excited atom.
The Effect of Collisions on a Lorentzian.

So far we have seen that

\[ \Delta P_{\text{homogeneous}} = \frac{1}{2\pi} \left( A_2 + k_2 \right) \]

where \( k_2 \) is the inelastic collision rate (collision between free electron and bound electron involving energy transfer).

But what happens if there are, in addition to free electrons, other atoms or molecules? Collisions between atoms or molecules are inelastic because little energy is exchanged in such collisions and the quantum state of the collision partners is unaffected. However, there is energy broadening of the energy levels while the collision is taking place which may take \( 10^{-13} \) s or less.

Overlap of electron cloud leading to broadening

\[ T_{\text{collision}} \approx \frac{10^{-10} \text{m}}{2 \times 10^3 \text{m/s}} \quad (v = \text{proton}) \]

\[ \approx 0.5 \times 10^{-13} \text{ s}. \]
But one cycle of $\lambda = 10 \mu m$ light is $\frac{\lambda}{c} \sim 3 \times 10^{-15}$, i.e. $2 \times 10^{-14}$ s.

So a collision can last 10 or 100 cycles and smearing of the energy levels can produce a phase broadening. We call this pressure or collisional broadening because it is proportional to the pressure of gas. Now

$$\Delta V \sim \frac{1}{2\pi} \left[ \left( n_2 + k_2 \right) + 2 D_{coll} \right]$$

where $D_{coll}$ is the collision frequency

$$D_{coll} = N \bar{\sigma}_{coll} \langle V \rangle$$ mean velocity

$\bar{\sigma}_{coll}$ collision cross section

$\langle V \rangle = \left[ \frac{8kT}{\pi} \left( \frac{1}{M_m} + \frac{1}{M_n} \right) \right]$ density of gas

$m$ and $n$ are the two species colliding

$N = 2 \times 10^{16}$ Pressure [torr]

Collisional broadening is also homogeneous broadening.