OPTICAL RESONANT CAVITIES

An optical cavity not only determines the transverse modes but the frequencies associated with these modes.

This is because an optical cavity has resonances at certain frequencies: longitudinal modes.

To understand the origin of these resonant frequencies consider an optical cavity made by two plane parallel reflecting surfaces separated by a distance \( d \).

If \( d = n \lambda/2 \) there is a stable standing wave pattern of the electric field between the two reflectors.
 Rewriting this

\[ n \lambda_1 = 2d \]

The frequency corresponding to this wavelength

\[ \nu_1 = \frac{nc}{2d} \]

This is just one of the resonant frequencies of the cavity. The next resonance has \((n+1)\) half wavelengths inside the cavity: hence

\[ \nu_2 = \frac{(n+1)c}{2d} \]

Thus the frequency separation between the resonant modes: **Free Spectral Range**

\[ \Delta \nu = \nu_2 - \nu_1 = \frac{c}{2d} \quad (FSR) \]

Clearly an optical cavity has an infinite number of these resonant modes.
Resonances of a Fabry-Perot Etalon

Free Spectral Range.

We call two parallel-plane mirrors a Fabry-Perot Etalon (Frequency Standard - French)

If a broadband radiation is either generated inside the etalon or injected into the etalon from the outside, the transmitted radiation is severely modulated with transmission maxima at these resonant frequencies. Why does this happen?

Minor 1  Minor 2

\[
\begin{align*}
\text{Incident Radiation} \quad & E_\Phi \\
\text{Reflectivity} \quad & \rho_1 \\
\text{Transmitted Radiation} \quad & E_T(\omega)
\end{align*}
\]
In the "etalon" configuration each mirror has multiple waves incident upon it. Consider for instance mirror 1.

\[
\begin{align*}
E_i & \quad \text{incident waves electric field} \\
E_T^+ & \quad \text{fraction of the incident wave's field that is transmitted and going in the} + z \text{ direction} + \text{the circulating electric field that is building up inside the cavity} \text{ propagating in} + z \text{ direction} \\
E_T^- & \quad \text{The amplitude of the circulating electric field that is incident upon mirror 1 in the} - z \text{ direction} \\
E_r & \quad \text{Total reflected electric field, has two parts. Fraction of the incident wave reflected by mirror 1} + \text{Fraction of the circulating wave transmitted} \\
\end{align*}
\]
How are these fields related to one another?

\[
\begin{bmatrix}
E_r \\
E_T^+
\end{bmatrix} =
\begin{bmatrix}
\rho & j\tau_1 \\
j\tau_1 & \rho
\end{bmatrix}
\begin{bmatrix}
E_i \\
E_T^-
\end{bmatrix}
\]

Fields leaving the mirror

Fields incident upon the mirror

\(\rho\) is the field reflectivity

\(\tau\) is the field transmission

\(\rho^2 + \tau^2 = 1\)  Conservation of energy / photons

\[\begin{align*}
E_r &= \rho, E_i + j\tau_1, E_T^- \\
E_T^+ &= j\tau_1, E_i + \rho, E_T^-
\end{align*}\]

Let's calculate what \(E_T^+\) is. To do this we need to do some bookkeeping. Let's assume that at any one instant the circulating field in the +z direction is the sum of the incident field that is transmitted \((j\tau_1 E_i = E_0)\) plus fraction of this transmitted field that has survived \(1, 2, 3 \ldots n\) bounces inside the etalon cavity.
\[ E_T^+ = E_0 + E_0 \rho_1 \rho_2 e^{-j \phi} (2nd) + E_0 (\rho_1 \rho_2) e^{2 - j \phi 2k (2nd)} + \ldots + E_0 (\rho_1 \rho_2) e^{m - j \phi m k (2nd)} \]

writing \( e^{-j k 2nd} = e^{-j \phi} \) the phase factor in traversing the distance 2nd (\( n \) refractive index of material between the two minn.)

\[ E_T^+ = E_0 \left( 1 + \rho_1 \rho_2 e^{-j \phi} + (\rho_1 \rho_2)^2 e^{-2j \phi} + \ldots + (\rho_1 \rho_2)^m e^{-jm \phi} \right) \]

This is a geometric series

\[ S = a + ar + ar^2 + \ldots + ar^n = a / (1 - r) \quad -1 < r < 1 \]

where \( r = \rho_1 \rho_2 e^{-j \phi} \)

\[ E_T^+ = E_0 / 1 - \rho_1 \rho_2 e^{-j \phi} \]
Then \( E_T^- \) must be

\[
E_T^- = E_T^+ p_2 e^{-j\phi}
\]

The net reflection of mirror 1

\[
\rho_{\text{net}} = \frac{E_r}{E_i} = \frac{\rho E_i + jE_e t_1}{E_i} = \rho_1 + t_1 E_e E_e^- E_i
\]

\[
\rho_{\text{net}} = \rho_1 + j t_1 \frac{E_e^+ p_2 e^{j\phi}}{E_i}
\]

\[
= \rho_1 + j t_1 \frac{E_0 e^{-j\phi}}{E_i (1 - \rho_1 p_2 e^{-j\phi})}
\]

but \( E_0 = j t_1 E_i \)

\[
= \rho_1 - t_1^2 p_2 e^{-j\phi} \frac{1}{1 - \rho_1 p_2 e^{-j\phi}} = \rho_1 + \frac{\rho_2 (\rho_1 - 1) p_2 e^{-j\phi}}{1 - \rho_1 p_2 e^{-j\phi}}
\]

\[
= \rho_1 - \rho_1^2 p_2 e^{-j\phi} + \rho_1^2 p_2 e^{-j\phi} - p_2 e^{-j\phi}
\]

\[
\rho_{\text{net}} = \frac{\rho_1 - p_2 e^{-j\phi}}{1 - \rho_1 p_2 e^{-j\phi}}
\]
The net transmission through the etalon

\[ T = 1 - \left| \text{net} \right|^2 \]

\[ = 1 - \left( \frac{p_1 - p_2 e^{-j\phi}}{1 - p_1 p_2 e^{-j\phi}} \right) \left( \frac{p_1 - p_2 e^{+j\phi}}{1 - p_1 p_2 e^{+j\phi}} \right) \]

\[ = 1 - \frac{p_1^2 - p_1 p_2 (e^{+j\phi} + e^{-j\phi}) + p_2^2}{1 - p_1 p_2 (e^{+j\phi} + e^{-j\phi}) + p_1^2 p_2^2} \]

But \( \cos 2\theta = e^{j\phi} + e^{-j\phi} \)

\[ T = 1 - \frac{p_1^2 - 2 p_1 p_2 \cos \phi + p_2^2}{1 - 2 p_1 p_2 \cos \phi + p_1^2 p_2^2} \]

\[ = 1 - \frac{p_1^2 - 2 p_1 p_2 \cos 2(nkd) + p_2^2}{1 - 2 p_1 p_2 \cos 2(nkd) + p_1^2 p_2^2} \]

\[ = 1 - \frac{p_1^2 - 2 p_1 p_2 \cos 2\theta + p_2^2}{1 - 2 p_1 p_2 \cos 2\theta + p_1^2 p_2^2} \]

But \( R_{1,2} = \frac{p_1}{p_{1,2}} \) and \( \cos 2\theta = 1 - 2\sin^2\theta \)
\[
T = \frac{1 - R_1 - 2\sqrt{R_1 R_2} \left( 1 - 2\sin^2 \theta \right) + R_2}{1 - 2\sqrt{R_1 R_2} \left( 1 - 2\sin^2 \theta \right) + R_1 R_2}
\]

\[
T = 1 - 2\sqrt{R_1 R_2} \left( 1 - 2\sin^2 \theta \right) + R_1 R_2 - R_1 + 2\sqrt{R_1 R_2} \times \\
\left( 1 - 2\sin^2 \theta \right) + (-R_2)
\]

\[
1 - 2\sqrt{R_1 R_2} \left( 1 - 2\sin^2 \theta \right) + R_1 R_2
\]

\[
= \frac{1 + R_1 R_2 - R_1 - R_2}{1 - 2\sqrt{R_1 R_2} \left( 1 - 2\sin^2 \theta \right) + R_1 R_2}
\]

\[
= \frac{(1 - R_1)(1 - R_2)}{\left[ 1 - (R_1 R_2)^{1/2} \right]^2 + 4 (R_1 R_2)^{1/2} \sin^2 \theta}
\]

\[\text{QED} \]