We saw that a Gaussian optical beam $\omega > \lambda$ spreads very slowly.

But how is such a beam produced?

In a laser oscillator, the feedback cavity determines the transverse amplitude and phase variation (the transverse mode) of the output beam.

The laser cavity also selects which frequencies will oscillate by providing a frequency-dependent feedback. These are the longitudinal modes of the cavity.

![光学共振器图示](image)

- $p < 1$
- $R_2 \sim \infty$
- Laser cavity without the gain medium
Stability of Optical Cavities.

How are the radii of curvature of the two mirrors chosen?

For stable feedback we would like to confine the rays that are bouncing between the mirrors.

The above cavity is clearly unstable.

For stable oscillation we want the height of the ray at any position within the cavity to be confined to a value between some maximum and minimum value that is smaller than the limiting aperture at that position.
Use Ray Matrix Technique to Derive a Condition for determining stability of Optical Cavities.

What is a ray matrix

\[
\begin{bmatrix}
    r \\
    r'
\end{bmatrix}_{\text{output}} =
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\begin{bmatrix}
    r \\
    r'
\end{bmatrix}_{\text{input}}
\]

The slope \( r' \) and the height \( r \) of a ray at any arbitrary output plane is given in terms of \( r' \) and \( r \) at the input plane by

\[
\begin{align*}
    r & = A r_i + B r'_i \\
    r_0 & = C r_i + D r'_i
\end{align*}
\]

Where \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is the ray matrix of all the space between the output and the input planes.
Example

Free Space Propagation

Clearly the slope \( r_0' = r_i' \)

The height \( r_0 = r_i + d r_i' \)

but

\[
\begin{bmatrix}
  r_0 \\
r_0'
\end{bmatrix} = \begin{bmatrix} A & B \\ c & D \end{bmatrix} \begin{bmatrix} r_i \\
r_i'
\end{bmatrix}
\]

\( r_0 = A r_i + B r_i' \)
\( = (1 \| r_i + (d) r_i' \)

\( r_0' = c r_i + D r_i' \)
\( = (0) r_i + (1) r_i' \)

\[
\begin{bmatrix} A & B \\ c & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad \text{Free Space}
**Thin Lens:** focal length \( f \)

\[
\begin{align*}
  r_0' &= -\left( \frac{r_i' - f r_i''}{f} \right) \\
  &= r_i'' - r_i/f, \\
  \begin{bmatrix} r_0' \\ r_{0'}' \end{bmatrix} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_i \\ r_i' \end{bmatrix} \\
  r_0 &= A r_i + B r_i' \\
  &= (1) r_i + (0) r_i' \\
  r_{0'} &= C r_i + D r_i' \\
  &= (-1/f) r_i + (1) r_i' \\
  \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ thin lens} &= \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}
\end{align*}
\]
Ray Matrix of a Compound System

Example:  Free Space, $d$, followed by a thin lens $f$

\[
\begin{bmatrix}
  r_0 \\
  r_0'
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 \\
  -\frac{1}{f} & 1
\end{bmatrix}
\begin{bmatrix}
  1 & d \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  r_i \\
  r_i'
\end{bmatrix}
\]

Lens  Free Space  
$f$  $d$  

ie we multiply the matrices of the individual elements. The order of the multiplication is important!

\[
= \begin{bmatrix}
  1 & d \\
  -\frac{1}{f} & (1-d/f)
\end{bmatrix}
= \begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\]
Now think of a laser cavity made up of two mirrors that are separated by a distance, d.

We can show that the radius of curvature $R$ of a mirror is related to its focal length by

$$R = 2f$$

A ray bouncing back and forth inside the cavity sees a biprismatic lens system.

Starting point $d$ $R_1 = 2f_1$ $R_2 = 2f_2$ $f_1$ $f_2$ $f_1$ $f_2$ $f_1$ $d$ $d$ $d$ $d$ $d$ so on...
Ray Matrix for a Bi-Periodic Lens System.

Let's consider the path taken by a ray during one round trip through the cavity.

\[
\begin{pmatrix}
  R_{S+1} \\
  R'_{S+1}
\end{pmatrix}
= \begin{pmatrix}
  1 & d \\
  -1/f_1 & 1-d/f_1
\end{pmatrix}
\begin{pmatrix}
  1 & d \\
  -1/f_2 & 1-d/f_2
\end{pmatrix}
\begin{pmatrix}
  R_s \\
  R'_s
\end{pmatrix}
\]

\[
\begin{pmatrix}
  r_{S+1} \\
  r'_{S+1}
\end{pmatrix}
= \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  r_s \\
  r'_s
\end{pmatrix}
\]

\[A = 1 - d/f_2\]
\[B = d \left(2 - d/f_2\right)\]
\[C = -\left[1/f_1 + 1/f_2 \left(1-d/f_1\right)\right]\]
\[D = -\left[d/f_1 - \left(1-d/f_1\right) \left(1-d/f_2\right)\right]\]

Where \(AD - BC = 1\)
Using algebraic notation

\[ r_{st1} = A r_s + B r_s' \]  (1)

\[ r_{st1}' = C r_s + D r_s' \]  (2)

\[ r_s' = \frac{1}{B} \left( r_{st1} - A r_s \right) \] but \( S \) is arbitrary

\[ r_{st1}' = \frac{1}{B} \left( r_{st2} - A r_{st1} \right) \]

Substituting in (2) for \( r_{st1}' \) and \( r_s' \)

\[ \frac{1}{B} \left( r_{st2} - A r_{st1} \right) = C r_s + \frac{D}{B} \left( r_{st1} - A r_s \right) \]  (3)

Now we have an equation for height of the ray at the output of plane \( st2 \) in terms of height of the ray at the output of plane \( st1 \) and the input \( S \).

Rearranging (3)

\[ r_{st2} - (A+D) r_{st1} + (AD-BC) r_s = 0 \]

\[ r_{st2} - 2b r_{st1} + r_s = 0 \]

where \[ 2b = (A+D) \]
For a stable cavity we would like to have the height of the ray to oscillate between some maximum and minimum value, \( r_{\text{max}} \) and \( r_{\text{min}} \).

\[
(\phi) = r_0 e^{i \phi}
\]

\[
r_{st_2} - 2b r_{st_1} + r_s = 0
\]

\[
2j\phi - 2be^{j\phi} + 1 = 0 \quad \text{solve quadratic}\]

\[
e^{-j\phi} = b \pm i \sqrt{1 - b^2}
\]

For \( \phi \) to be real we want \( |b| \leq 1 \).

but \( b = (A + D)/2 \)

\[-1 \leq (A + D)/2 \leq +1 \quad \text{Sub for } A \text{ and } D\]

\[-1 \leq 1 - d/f_2 - d/f_1 + d^2/(2f_1f_2) \leq 1 \quad \text{add 1 and divide by 2}
\]

\[
0 \leq \left(1 - \frac{d}{2f_1}\right) \left(1 - \frac{d}{2f_2}\right) \leq 1
\]

\[
0 \leq g_1, g_2 \leq 1 \quad \text{Stability criterion.}
\]
Confocal \( (g_1 = g_2 = 0) \), plane-plane \( (g_1 = g_2 = 1) \) and spherical \( (g_1 = g_2 = -1) \) are all on the borderline of stability. Most lasers avoid this by increasing or decreasing \( d \) slightly.
The Unstable Resonator (Cavity)

When the stability condition is not obeyed, the rays walkoff. This does not mean that an unstable cavities are not used in lasers.

In a very high gain, short lived laser (eg excimer, KrF, XeCl, ArCl)

an unstable cavity is used to extract energy from the medium.

Convex mirror

Gain Medium

Annular output laser beam.