a) Find an expression for the resonant frequencies of the TEM\(_{0,0}\) modes of the cavity.

A gaussian mode TEM\(_{0,0}\) is sustained by the cavity, with longitudinal phase term given by

\[
\varphi(z) = (kz - \tan^{-1}(z/d_0))
\]

such that at \(z=0\) \(\varphi(0)=0\)

\[
z = d \rightarrow \varphi(d) = kd - \tan^{-1}(d/d_0)
\]

After a round trip, the resonant condition is:

**Round Trip Phase Shift** = \(q \cdot 2\pi\) = \(2(\varphi(d) - \varphi(0))\) = \(2(kd - \tan^{-1}(d/d_0))\)

\(q\): integer

Thus \(kd - \tan^{-1}(d/d_0) = \pi q \rightarrow \frac{2\pi q}{c} d = \pi q + \tan^{-1}(d/d_0)\)

\(q\): integer

\[\nu_q = q\text{-th resonant frequency} = \frac{c}{2kd} \left( q + \frac{1}{\pi} \tan^{-1}\left(\frac{d}{d_0}\right) \right)\]

Now we need to calculate \(d_0\), and since the radius of curvature of the mirrors must match those of the gaussian beam, we have:

\[R(d) = d \left( 1 + \left(\frac{z_0}{d}\right)^2 \right) = R_2 \rightarrow z_0 = d \sqrt{\frac{R_2}{c}} = \frac{3R_2}{c} \frac{1}{V_3}\]

and

\[
\nu_q = \frac{2c}{3R_2} \left( q + \frac{1}{\pi} \tan^{-1}\left( \frac{3R_2}{4} \left( \frac{1}{18} R_2 \right) \right) \right) = \frac{2c}{3R_2} \left( q + \frac{1}{\pi} \tan^{-1}(1/3) \right) = \frac{2c}{3R_2} \left( q + \frac{1}{3} \right)
\]
b) If \( R_2 = 2 \text{m} \) and \( \lambda_0 = 5000 \text{ Å} \) (wavelength region of interest), the free spectral range in MHz and Å units: FSR

\[
\text{FSR} = \frac{2\pi c}{\lambda_0} = \frac{2\pi c}{3R_2} \left( \frac{1}{\lambda_0} + \frac{1}{3} \right) = \frac{2\pi c}{3R_2} \left( \frac{1}{\lambda_0} + \frac{1}{3} \right)
\]

\[
= \frac{2.3 \times 10^8}{3.2} = 100 \text{ MHz}
\]

Using \( \frac{\Delta \lambda}{\lambda_0} = \frac{\Delta \nu}{\nu_0} \), the FSR (Å) = FSR (Hz) \times \frac{\lambda_0}{\nu_0}

\[
\text{FSR (Å)} = \left( \frac{100 \times 10^6}{3 \times 10^8} \right) \frac{5000 \times 10^6}{0.83 \times 10^{-3}} = 0.83 \times 10^3 \text{ Å}
\]

2) Quality factor of the cavity: \( Q \)

\[
Q = \frac{\nu_0}{\Delta \nu/2} = \frac{\nu_0}{\Delta \nu_{1/2}}
\]

since we are interested in the spectral region around \( \nu_0 = c/\lambda_0 \)

and \( \Delta \nu_{1/2} = \frac{\lambda_0}{2c} \left[ \frac{1 - \sqrt{T_1^2 + T_2^2}}{1 - \sqrt{T_1^2 T_2^2}} \right]^{1/4} \)

Thus

\[
Q = \frac{\nu_0}{\lambda_0} \left[ \frac{1 - \sqrt{T_1^2 + T_2^2}}{1 - \sqrt{T_1^2 T_2^2}} \right]^{1/4} \]

\[
= \frac{2.3 \times 10^8}{3 \times 10^8} \left[ \frac{1 - \sqrt{0.99 \times 0.97}}{1 - \sqrt{0.99 \times 0.97}} \right]^{1/4}
\]

\[
= 0.93 \times 10^9
\]

3) Finesse: \( F \)

\[
F = \frac{\text{FSR}}{\Delta \nu_{1/2}}
\]

with \( \Delta \nu_{1/2} = \frac{2\pi c}{3R_2} \left( \frac{1 - \sqrt{0.99 \times 0.97}}{1 - \sqrt{0.99 \times 0.97}} \right)^{1/4} = 0.645 \text{ MHz} \)

Thus

\[
F = \frac{100 \text{ MHz}}{0.645 \text{ MHz}} = 155.1
\]
4) Photon lifetime: \( T_p \)

\[
T_p = \frac{\text{Round trip}}{1 - S}
\]

where \( S = \text{survival factor after one round trip} \)

Reflectivity: \( T_1^2 = 0.99 \), \( T_2^2 = 0.97 \)

After one round trip the survival factor for this cavity is

\[
S = T_1^2 T_2^2
\]

\[
T_p = \frac{2 d / c}{1 - T_1^2 T_2^2} = \frac{3}{1 - (0.99 \times 0.97)} = 251.84 \text{ ns}
\]

\[6.2\]

\[ R_1 = 0.99 \]

\[ R_3 = 0.99 \]

\[ R_2 = 0.99 \]

where \( R_i \) refers to the reflecting in this problem, that is

\[ R_i = T_i^2 \]

what is the photon lifetime?

Round trip time = \( T_{RT} = \frac{2 (d_1 + d_2)}{c} = \frac{3 \text{m}}{3 \times 10^8 \text{m/s}} = 10 \text{ nsec} \)

Survival factor = \( R_1 R_3 R_2 R_1 \)

\[
T_p = \frac{T_{RT}}{1 - R_1 R_3 R_2 R_1} = \frac{10 \text{ nsec}}{1 - (0.99)^3 \times 0.9} = 78.9 \text{ nsec}
\]

\[6.3\] what is the cavity \( Q \)? (Assuming the wavelength region of interest is 5000 Å)

Given the photon life-time, from equation (6.4.5): \( \Delta \omega / \omega = \frac{1}{T_p} \)
and from equation (6.3.5)

\[ Q = \frac{V_1}{\Delta V_2} = \frac{(C/\Delta V_2)}{2\pi \frac{\Delta V_2}{2\pi}} = \frac{C \cdot 2\pi}{\Delta V_2} = \frac{(3 \times 10^8)(78.9 \times 10^{-9})}{5000 \times 10^{-10}} \]

\[ Q = 2.97 \times 10^8 \]

6.4. If path 3 has a transmission coefficient \( T_3 = 0.85 \), then:

\[ d_2 = 0.5 \text{m} \]

a) Photon lifetime:

The survival factor now becomes: \( S = T_4 \cdot R_1 \cdot R_2 \cdot R_3 \cdot R_4 \)

Thus,

\[ T_P = \frac{T_{\text{TOT}}}{1 - S} = \frac{10 \text{ nsec}}{1 - 0.99 \times 0.9 \times 0.85} = 338 \text{ nsec} \]

b) Suppose path 4 had a power gain of \( G_4 = 1.1 \). What is the new photon lifetime?

\[ T_4 = G_4 = 1.1 \quad \Rightarrow \quad T_P = \frac{10 \text{ nsec}}{1 - (0.99)^3 \times 0.9 \times 1.1} = 253.7 \text{ nsec} \]

c) If we blindly plug into the formulas, \( T_P \) becomes negative for \( G \) sufficiently large. What is the meaning of this apparent absurdity?

If \( T_P < 0 \) then the rate of change of photons with time in the cavity is positive:

\[ \frac{dN_P}{dt} = \frac{N_P}{T_P} = \frac{N_P}{|T_P|} > 0 \]

which indicates that the number of photons is growing with time.
6.5 The optical cavity of the diagram is excited by a variable-frequency source, and the detected intensity is as shown:

![Intensity diagram](image)

What is the nominal wavelength?

Since $\text{FSR} = 125 \times 10^6 \text{ Hz} << \nu_0 = 5 \times 10^{14} \text{ Hz}$, the resonant frequencies are close enough to each other to take the nominal frequency as $\nu_0 \approx \nu_0 = 5 \times 10^{14} \text{ Hz}$ so the nominal wavelength: $\lambda_0 = c/\nu_0 = 0.6 \text{ \mu m}$

6.6 How long is the cavity?

From $\text{FSR} = \frac{c/(n \cdot \lambda_0)}{2d}$ with $n = 1$ (refractive index) for free space

$$125 \text{ MHz} = \frac{3 \times 10^8 \text{ m/s}}{2d} \Rightarrow d = 1.2 \text{ m}$$

6.7 Finesse? From the graph: $\Delta \nu_1/2 = 2.5 \text{ MHz}$

$$F = \frac{\text{FSR}}{\Delta \nu_1/2} = \frac{125 \text{ MHz}}{2.5 \times 10^6} = 50$$

6.8 What is $Q$?

$$Q = \frac{\nu_0}{\Delta \nu_1/2} = \frac{5 \times 10^{14}}{2.5 \times 10^6} = 0.2 \times 10^8$$

6.9 What is the photon lifetime?

Using $\tau_p \cdot \Delta \nu_1/2 = 1 \Rightarrow \tau_p = \frac{1}{2 \pi \Delta \nu_1/2} = 63.6 \text{ nsec}$

$$\tau_p = \frac{2d/c}{1 - R_1 R_2} \Rightarrow R_1 R_2 = 0.874$$

$$G^2 R_1 R_2 = 1 \Rightarrow G = 1.069$$
6.11) Given the following cavity and a nominal wavelength: \( \lambda_0 = 0.6328 \mu m \)

\[
\begin{align*}
T_1^2 &= R_1 = 0.985 \\
d &= 50 cm \\
T_2^2 &= R_2 = 0.97 \\
T_3^2 &= R_3 = 0.97
\end{align*}
\]

Scattering from lens surface = 1%/pass

On each of the lens surfaces there is a 1% intensity loss and a 99% transmission. The transmission coefficient is therefore:

\[ T_{\text{surface}} = (1 - 0.01) = 0.99 \text{ through each surface} \]

\[ T_{\text{total}} = (0.99)^2 \text{ through the lens} \]

a) Photon lifetime:

round trip time = \( \frac{3d}{c} = 5 \text{ nsec} \)

survival factor = \( R_1 R_2 R_3 T_{\text{total}} \)

\[ T_p = \frac{T_{\text{total}}}{1 - 5} = \frac{5 \text{ nsec}}{1 - (0.97)^3} = 0.985 \times (0.99)^2 = 54.55 \text{ nsec} \]

b) Cavity Q?

From \( Q = \frac{\lambda_0}{\Delta \lambda} = \frac{c}{\lambda_0} \times T_p = \frac{3 \times 10^8}{0.6328 \times 10^{-6} 	imes 2 \times (54.55 \times 10^{-9})} = 1.62 \times 10^8 \)

NOTE: THAT THE RESULTS PROVIDED IN THE

TEXT BOOK ARE WRONG.
Drawn to scale on the graph below is the relative power transmission through a Fabry-Perot cavity, when the distance \( d_0 \) is increased slightly. The source is a He:Ne laser at \( \lambda_0 = 6328 \) Å.

**Dimensions measured on the graph:** 
- \( \delta d = 3.1 \) cm
- \( \Delta d \frac{1}{2} = 0.5 \) cm

**a) What is the distance \( d_0 \)?**

Resonance occurs when 
\[
\frac{d}{d_0} = \frac{1}{2}
\]
Thus if \( d_0 = \frac{1}{2} \Delta d \), then
\[
(d_0 + \delta d) = (\frac{1}{2} + 1) \frac{1}{2} = \frac{3}{2}
\]
and 
\[
\delta d = \text{separation distance} = \frac{\Delta d}{2} = 3164 \text{ Å}
\]

The graph is drawn on scale, and when measured on the graph, the distance \( \delta d = 3.1 \) cm, thus the ratio at which the graph has been drawn is given by:

\[
\text{Ratio} = \frac{3164 \text{ Å}}{3.1 \text{ cm}} = 1.02 \times 10^{-5}
\]

**b) What is the finesse of the cavity?**

Tuning of the cavity is being done by adjusting the separation \( d \) between the mirrors. The transmitted intensity is given as a function of \( d \), so we need to determine the cavity parameters in terms of \( d_0 \) (separation distance between resonances) and \( \Delta d \frac{1}{2} \) (FWHM).

\[
V_\lambda = \frac{q}{2d_0} \rightarrow \Delta V_\lambda = \frac{q}{2} \left( \frac{\Delta d}{d_0^2} \right) \quad \text{and} \quad \Delta V_\lambda = \frac{V_\lambda}{2} \Delta d_0 \quad \text{or} \quad \frac{\Delta V_\lambda}{V_\lambda} = \frac{\Delta d_0}{d_0}
\]
Now: \[ \text{Finesse} = \frac{\text{FSR (Hz)}}{\Delta v} = \frac{\text{FSR} \cdot \Delta d_2}{\nu_0} = \frac{\text{FSR} \cdot \Delta d_2}{(9 \cdot \text{FSR})} = \frac{5d}{\Delta d_2} \]

where we used: \[ \frac{\Delta d}{\nu} = \frac{5d}{9} \text{ and } \nu_0 = 9 \cdot \text{FSR} \]

Thus \[ F = \frac{5d}{\Delta d_2} = \frac{3.4 \text{ cm} \times \text{Ratio}}{0.5 \text{ cm} \times \text{Ratio}} = 6.2 \]

(c) What is the cavity Q?

Similarly \[ Q = \frac{\nu_0}{\Delta v} = \frac{\nu_0}{\Delta d_2} = \frac{3.4 \text{ cm}}{0.5 \text{ cm} \times \text{Ratio}} = \frac{6.8 \times 10^{-2}}{5.1 \times 10^{-8}} = 0.196 \times 10^6 \]

6.13 Drawn to scale on the graph below is the relative power transmitted through the cavity as the distance d is increased from its initial value of 2 cm to 2 cm + 0.5 cm. The source is a single-mode laser of wavelength \( \lambda_0 \).

![Graph showing relative power transmission through the cavity with dimensions and ratios indicated.]

- Dimension on the graph = 6.8 cm
- Real dimension = 0.4 \( \mu \)m
  \[ \text{Ratio} = \frac{0.4 \times 10^{-6}}{6.8 \times 10^{-2}} = 5.88 \times 10^{-6} \]

The scaled distance 0.4 \( \mu \)m on the graph corresponds to 6.8 cm if we measure d. The ratio between the real and the measured dimensions is 5.88 \times 10^{-6}.

(a) Wavelength of the source?

Resonance occurs when \( d_0 = \frac{\lambda}{2} \), thus \( 5d = \frac{\lambda}{2} \)

On the graph \( 5d = 5.5 \text{ cm} \), then \( 5d = 5.5 \text{ cm} \times \text{Ratio} = 0.3234 \mu \text{m} = \frac{\lambda}{2} \)

\[ \lambda = 0.6468 \mu \text{m} \]
b) **Finesse?**

The measured width $\Delta y_2$ on the graph at the points where the intensity is $\frac{I_{\text{max}}}{2}$ is 0.9 cm. Then

$$\text{Finesse} = \frac{\Delta d_f}{\Delta y_2} = \frac{5.5 \text{ cm} \times \text{Ratio}}{0.9 \text{ cm} \times \text{Ratio}} = 6.11$$

d) **FWHM in MHz?**

$$\text{Finesse} = \frac{\text{FSR(Hz)}}{\Delta y_2} = \frac{c}{\lambda d_{fo}} \Rightarrow \Delta y_2 = \frac{c}{2d_{fo} F} = \frac{9 \times 10^8}{2 \times 2 \times 10^{-2} \times 6.11} = 1227.5 \text{ MHz}$$

c) **Q?**

$$Q = \frac{\nu_0}{\Delta y_2} = \frac{c/\lambda_{fo}}{\Delta y_2} = \frac{9 \times 10^8 / (0.6468 \times 10^{-6})}{1227.5 \times 10^6} = 3.778 \times 10^5$$

e) **$\tau_p$?**

$$\tau_p \cdot \Delta \omega_2 = \tau_p \cdot 2\pi \cdot \Delta y_2 = 1$$

$$\Rightarrow \tau_p = \frac{1}{2\pi \cdot 1227.5 \times 10^6} = 0.1296 \text{ nsec}$$