EE172L
Bragg Diffraction

Purpose
Study the interaction between optical waves and high frequency acoustic waves in Bragg diffraction.

Background
In the Polarization and Phase Modulation Lab, we studied the modulation of the phase across the entire laser beam by an acoustically induced index variation. The interaction was modeled as a time-dependent wave plate with a Jones matrix formulation. Such a formulation is valid in the low (acoustic) frequency region where the wavelength of sound, $\lambda_s$, is much larger than the spot size of the optical beam, $W$. In this lab, we will examine the high acoustic frequency region of operation where $\lambda_s \leq W$. Because the size of sound wave $\lambda_s$, the entire optical wave front no longer experiences the same index variation. A sinusoidal phase variation from the sound wave is imposed over the cross section of the optical beam, creating a phase grating which scatters the optical wave.

There are three different approaches that are commonly used to describe the interaction between the acoustic and electromagnetic field, as indicated in the textbook (Acousto-optics p601). Here we will describe an approach that combines the parametric process model and the coupled wave theory model. Starting with one of Maxwell’s equations.

$$\nabla^2 \vec{E} = \mu \frac{\partial^2 \vec{D}}{\partial t^2} = \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \frac{\partial^2 \vec{P}}{\partial t^2} \tag{1}$$

Where polarization, $\vec{P}$, is the driving term of the differential equation. This is a very general form used to couple the interaction between EM waves and materials or other forces into the wave equation. (Note that if $\vec{P} = 0$, Eq. 1 becomes the familiar wave equation for propagation in homogenous, isotropic, non-conducting medium). Now, let us assume that the most significant interaction is with a modulated acoustic field in the propagating medium. Define a change of polarization vector field, $\Delta \vec{P}_i$:

$$\Delta \vec{P}_i = -\frac{\varepsilon_d}{\varepsilon_o} \tilde{P}_{adi} \tilde{S}_{kl} \vec{E}_d$$

with a scattered electric field, $\vec{E}_d$, in the d direction + material strain component, $\tilde{S}_{kl}$, in the k and l direction caused by the sound field + (coupled through) material photoelastic tensor, $\tilde{P}_{adi}$, $\Rightarrow$ to produce polarization, $\Delta \vec{P}_i$, in the direction $i$.

Due to the nature of tensors and differential equations involved, the direction of the effect is reversible. In other words, the left and right side of Eq. 1 are both cause and effect for each other. Let’s define the form for electric fields on both sides:
input field: \[ E_i(\vec{r},t) = \frac{1}{2} E_i(r_i) \exp\left[j(\omega_i t - \vec{k}_i \cdot \vec{r})\right] + c.c \]
diffracted field: \[ E_d(\vec{r},t) = \frac{1}{2} E_d(r_d) \exp\left[j(\omega_d t - \vec{k}_d \cdot \vec{r})\right] + c.c \]

and the strained field caused by an acoustic wave with frequency, \( \omega_s \), propagating along the \( \vec{k}_s \) direction.
\[ S_{kl}(\vec{r},t) = \frac{1}{2} S_{kl} \exp\left[j(\omega_s t - \vec{k}_s \cdot \vec{r})\right] + c.c. \]

where \( c.c. = \) complex conjugate of the previous term

Now, expand the left side of Eq. 1,
\[
\nabla^2 \vec{E} = \frac{\partial^2 E_x}{\partial x^2} \vec{x} + \frac{\partial^2 E_y}{\partial y^2} \vec{y} + \frac{\partial^2 E_z}{\partial z^2} \vec{z} \\
= \vec{x} \left[ -\frac{1}{2} \frac{\partial}{\partial x} \left( j k_x E_x e^{-jk_x x} - \frac{\partial E_x}{\partial x} e^{-jk_x x} \right) \right] e^{j\omega t} \\
+ \vec{y} \left[ -\frac{1}{2} \frac{\partial}{\partial y} \left( j k_y E_y e^{-jk_y y} - \frac{\partial E_y}{\partial y} e^{-jk_y y} \right) \right] e^{j\omega t} \\
+ \vec{z} \left[ -\frac{1}{2} \frac{\partial}{\partial z} \left( j k_z E_z e^{-jk_z z} - \frac{\partial E_z}{\partial z} e^{-jk_z z} \right) \right] e^{j\omega t} \\
= \sum_{i=x,y,z} \vec{e} \left[ -\frac{1}{2} \left( k_i^2 E_i e^{-jk_i i} + 2 j k_i \frac{\partial E_i}{\partial i} e^{-jk_i i} - \frac{\partial^2 E_i}{\partial i^2} e^{-jk_i i} \right) \right] e^{j(\omega t - k_i \vec{r})} \\
\approx \sum_{i=x,y,z} \vec{e} \left[ -\frac{1}{2} \left( k_i^2 E_i + 2 j k_i \frac{\partial E_i}{\partial i} \right) \right] e^{j(\omega t - k_i \vec{r})} \tag{2}
\]

The 2nd derivative with respect space can be dropped using the slowly varying envelope approximation. Expand the polarization term, and we have:
\[ \Delta P_i(\vec{r},t) = -\frac{1}{4} \epsilon_i \epsilon_i' \vec{e}_i' \vec{P}_{idkl} E_d(r_d) \left\{ \exp\left[j(\omega_d t - \vec{k}_d \cdot \vec{r})\right] + c.c. \right\} S_{kl} \left\{ \exp\left[j(\omega t - \vec{k}_s \cdot \vec{r})\right] + c.c. \right\} \]
\[ = -\frac{1}{4} \epsilon_i \epsilon_i' \vec{P}_{idkl} E_d(r_d) S_{kl} \left\{ \exp\left[j((\omega_d t \pm \omega_s) t - (\vec{k}_d \pm \vec{k}_s) \cdot \vec{r})\right] + c.c. \right\} \]
\[ \epsilon_i' = \frac{\epsilon_i}{\epsilon_o} = n_i^2 \quad \text{and} \quad \epsilon_i' = \frac{\epsilon_d}{\epsilon_o} = n_d^2 \quad n = \text{index of refraction} \]

Put everything back into Eq. 1 yields:
\[ -\frac{1}{2} \left[ k_i^2 E_i + 2 j k_i \frac{\partial E_i}{\partial i} \right] e^{j(\omega t - k_i \vec{r})} \]
\[ E_i(\bar{r}_i) e^{j(\omega_i - \bar{k}_i \cdot \bar{r}_i)} \]

\[ + \frac{\mu}{4} \varepsilon \varepsilon' (\omega_d \pm \omega_s)^2 \tilde{P}_{ijkl} E_d(\bar{r}_d) S_{kl} \left[ \exp \left( j((\omega_d \pm \omega_s) t - (\bar{k}_d \pm \bar{k}_s) \cdot \bar{r}) \right) \right] \]  

(3)

Matching the terms on both sides of Eq. 3, we realize that in order to have non-zero solutions, the following relationship must exist:

\[ \omega_s = \omega_d \pm \omega_s \quad \text{physically, this satisfies energy conservation.} \]  

(4)

\[ \bar{k}_s = \bar{k}_d \pm \bar{k}_e \quad \text{physically, this satisfies momentum conservation.} \]  

(5)

Therefore, from the 2\text{nd} terms on both sides, we have

\[ \frac{\partial E_i}{\partial r_i} = \frac{j}{4} \omega \sqrt{\mu \varepsilon \varepsilon'} \varepsilon' \tilde{P}_{ijkl} E_d(\bar{r}_d) S_{kl} \exp \left( -j(\bar{k}_d \pm \bar{k}_s - \bar{k}_i) \right) \]  

(6)

Apply the following substitutions,

\[ k = \omega \sqrt{\mu \varepsilon} = \frac{2\pi}{\lambda} \quad \eta_{id} = \eta_{di} = \frac{\pi n^3}{2\lambda} \tilde{P}_{ijkl} S_{kl} \quad \varepsilon' = \varepsilon' = n^2 \]

We arrive at the following coupled wave equation:

\[ \frac{\partial \tilde{E}_i}{\partial r_i} = j \eta_{id} \tilde{E}_d \exp \left( j(\bar{k}_i - \bar{k}_d \pm \bar{k}_s) \cdot \bar{r} \right) \]  

(7)

Similarly, through the same operation with the diffracted field \( E_d \), we have:

\[ \frac{\partial \tilde{E}_d}{\partial r_d} = j \eta_{di} \tilde{E}_i \exp \left[ -j(\bar{k}_i - \bar{k}_d \pm \bar{k}_s) \cdot \bar{r} \right] \]  

(8)

Thus, depending on the resulting phase from the dot product, one field will grow at the expense of the other field. Energy of \( E_i \) can be transferred to \( E_d \) or vice versa through the interaction with the sound field. The equations also show that the energy transfer is most efficient when the three \( k \) vectors sum to zero. \( \bar{k}_s = \bar{k}_d \pm \bar{k}_e \) is called the phase matching condition. (see Fig. 1).

We can demonstrate the phase matching condition graphically. First, look at the magnitude of the three \( k \) vectors. With the sound frequency much lower than the light frequency, \( \omega_s << \omega_i \approx \omega_d \), the optical \( k \) vectors will have almost the same length.

\[ k_s = \omega_s / c \approx k_d = \omega_d / c \approx k. \]

However, because the speed of sound is much smaller than speed of light, the \( k \) vector of sound will be comparable in size to the optical \( k \) vector. Orienting the directions of \( k \) leads to the vector diagram for the phase matching condition.

Recall that \( k_s = \omega_s / \nu_s = 2\pi / \Lambda_s, k = 2\pi n / \lambda_s \)

where \( \Lambda = \text{acoustic wavelength}, \nu_s = \text{speed of sound}, n = \text{index of refraction} \)

From the diagram, an expression identical to the Bragg condition can be derived.

\[ k_s = (k_i + k_d) \sin \theta \approx 2k \sin \theta \]
\[ \lambda_s/n = 2\Lambda \sin \theta \] 

or as in Guenther (14-62) \( \sin \theta_g = \frac{k_s}{2k_i} \)

where the index variation created by the sound with spacing of \( \Lambda \) is similar to the atomic lattice planes that diffract X-rays. The major difference is that the moving sound field induces a shift in the optical frequency (Eq. 4) while the X-ray frequency remains the same after diffraction off stationary atomic lattice planes. The shift in frequency can also be viewed as the Doppler effect.

Now, assuming we have the phase matching condition, Eq. 7 and 8 can be solved with some change of variable manipulation. First, define \( \zeta \) along the bisector between the \( \vec{k}_i \) and \( \vec{k}_d \) vector direction. The projection of \( \zeta \) on the \( \vec{k}_i \) and \( \vec{k}_d \) vectors becomes \( r_i \) and \( r_d \).

Length of the projection: \( r_i = \zeta \cos(\theta) \) and \( r_d = \zeta \cos(\theta) \)

Rewrite Eq. 7 and 8 in terms of the projected length \( \zeta \) and angle \( \theta \).

\[
\frac{\partial E_i}{\partial \zeta} = \frac{\partial E_i}{\partial r_i} \cos(\theta) = i\eta E_d \cos(\theta) \quad (10)
\]

\[
\frac{\partial E_d}{\partial \zeta} = \frac{\partial E_d}{\partial r_d} \cos(\theta) = i\eta E_i \cos(\theta) \quad (11)
\]

The solution of the coupled equation is

\[
E_i(\zeta) = E_i(0) \cos(\eta \zeta \cos(\theta)) + iE_d(0) \sin(\eta \zeta \cos(\theta)) \quad (12)
\]

\[
E_d(\zeta) = E_d(0) \cos(\eta \zeta \cos(\theta)) + iE_i(0) \sin(\eta \zeta \cos(\theta)) \quad (13)
\]

After changing variables back to the original parameters,

\[
E_i(r_i) = E_i(0) \cos(\eta r_i) + iE_d(0) \sin(\eta \zeta r_i) \quad (14)
\]

\[
E_d(r_d) = E_d(0) \cos(\eta r_d) + iE_i(0) \sin(\eta \zeta r_d) \quad (15)
\]

With initial condition of \( E_d(0) = 0 \) (no diffracted field to start with)

\[
E_i(r_i) = E_i(0) \cos(\eta r_i) \quad (16)
\]

\[
E_d(r_d) = iE_i(0) \sin(\eta \zeta r_d) \quad (17)
\]

For the energy coupling process to occur, all three fields (\( E_i \), \( E_d \), and the acoustic wave) must overlap (be at the same location). Since \( \vec{k}_i \) and \( \vec{k}_d \) vectors are not parallel, they will...
eventually “walk-off” from each other. The finite distance in which they overlap is called the interaction length $L$. We can now write the expression for diffraction efficiency $\eta_{\text{diff}}$.

$$\eta_{\text{diff}} = \frac{I_{\text{diffracted}}}{I_{\text{incident}}} = \frac{E_{\text{diffracted}}^2}{E_{\text{incident}}(0)} = \sin^2(\eta L) = \sin^2\left(\frac{\pi n^3}{2\lambda} p SL\right)$$ \quad (18)

$$pS \equiv p_{\text{ad}}S_{\text{kl}} \quad S = \sqrt{\frac{2I_{\text{acoustic}}}{\rho v_s^3}} \quad \rho = \text{density}(\text{kg/m}^3), \quad v_s = \text{speed of sound} (\text{m/sec})$$

We can define an acoustic figure of merit $M_2$ (App. 14-C)

$$M_2 = \frac{n^6 \rho^2}{\rho v_s^3} \quad \text{Note that } M_2 \text{ is proportional to index } n \text{ to the 6th power.}$$

Finally, the equation for diffracted wave amplitude as a function of acoustic power is

$$\frac{I_{\text{diffracted}}}{I_{\text{incident}}} = \sin^2\left(\frac{\pi L}{\sqrt{2\lambda}} \sqrt{\frac{n^6 \rho^2}{\rho v_s^3} I_{\text{acoustic}}} \right) = \sin^2\left(\frac{\pi L}{\sqrt{2\lambda}} \sqrt{M_2 I_{\text{acoustic}}} \right) = \sin^2\left(\frac{\pi L}{\sqrt{2\lambda}} \Delta n \right)$$ \quad (19)

The formulation above describes the diffraction of incident light being most efficient in the direction that satisfies the Bragg condition (Eq. 9), with diffraction efficiency given by Eq. 19. As we can see, given sufficient interaction length $L$ or acoustic intensity, it is possible to achieve 100% diffraction efficiency.

The energy and momentum conservation theory is adequate in explaining the first order diffraction at angle $\pm \theta$. However, no higher order diffraction is predicted (diffraction at angle $\pm 2\theta$, $\pm 3\theta$, $\pm 4\theta$, ...), yet they are observed in the experiment (Figure 14B-3, p628 in Guenther). The problem lies in the fact that the acoustic wave front is not an infinite plane wave in real life situation. If the acoustic wave width is small, it acts like a thin phase grating. When an optical beam passes through this phase grating, a sinusoidal phase variation is imposed across the cross-section of the beam. At the far field, light is diffracted in the direction where the phase adds up constructively. Such diffraction is called Raman-Nath scattering and it can be modeled with the formulation developed for thin phase holograms (Appendix 12-A, Guenther). To predict the angle of the higher order diffraction, we can modify the $k$ vector phase matching diagram so that the optical $k$ vectors matches multiples of the acoustic $k$ vector. For example, $l^{th}$ order diffraction will occur at the angle $\theta_l$ where $\vec{k}_i$ and $\vec{k}_d$ phase match with the $l$ number of $\vec{k}_s$,

$$\sin \theta_l = \frac{l \cdot k_s}{2k_i} \quad \text{or} \quad \frac{\lambda_s}{n} = \frac{2\lambda}{l} \sin \theta_l$$ \quad (20)

The diffraction efficiency is a much more complex expression.
\[ \frac{I_i}{I_o} = J_l^2 \left( \frac{\Delta n k_s L}{n2\cos(\theta_i)} \right) = J_l^2 \left( \frac{k_s L}{n2\cos(\theta_i)} \sqrt{M_1 I_{\text{acoustic}}} \right) \]  

(21)

where \( J_l \) is the \( l^{th} \) order Bessel function. (In this case, \( L \) is just the width of the acoustic wave because \( \tilde{k}_s \) and \( \tilde{k}_d \) overlapped within such small distance.) If we keep all the parameters constant and only increase the acoustic intensity, the diffraction of energy from the 0\( ^{th} \) order beam (undiffracted incident beam) to the higher order beam will follow the square of the \( l^{th} \) order Bessel functions. This is also shown in Fig. 14B-3 (p628) in the textbook.

In summary, there are two sub-regions within the high acoustic frequency region,
1. Infinite plane acoustic wave front = First order Bragg diffraction
2. Small acoustic wave width (thin phase grating) = multiple order Raman-Nath Scattering

In this Lab, because the device we use has acoustic wave width = 5 mm, it operates near the Raman-Nath scattering region. We should expect to see higher order diffraction.

**Pre-Lab Reading**: “Modern Optics” by R. Guenther: Acoustooptics (p601-614), Appendix 14-B (p620-629), Appendix 12-A, phase hologram (p512-514), Appendix 14-C (p630-p631)

**Pre-Lab Exercise**:
1. Show mathematically that the angle of diffraction increases when the acoustic frequency is increased with constant laser frequency.
2. Show mathematically that the angle of diffraction decreases when the laser frequency increases with constant acoustic frequency.
3. A red He-Ne laser beam is Bragg diffracted by an acoustic modulator operating at 100 MHz. Assume that the modulator material is quartz glass with speed of sound = 5.97 km/sec, index of refraction = 1.46. Calculate the Bragg diffraction angle inside the quartz glass. Using Snell’s law, calculate the diffraction angle as seen from outside (air). What’s the frequency shift on the Bragg diffracted beam? If the diffraction is in the Raman-Nath region, what would the outside angles for the 1\( ^{st} \), 2\( ^{nd} \) and 3\( ^{rd} \) order diffraction spots?

\[ \theta_1 = \text{inside angle} \quad \theta_2 = \text{outside angle} \]
Experimental Procedure:

**NOTE:** Be Very Gentle in handling all the HeNe laser tubes. The glass tube inside can be destroyed from shock force. Do not drop, or pile them up on top of each other. Do not tighten the holding screw with more than gentle finger force.

1. Set the angle of incidence to be $0^\circ$. By retro-reflecting the beam off the modulator input window back into the laser or rotating the modulator until the diffraction pattern is symmetric. Use the red HeNe laser (632.8 nm) for this part. Be sure that the laser beam is not “cut-off” by the aperture of the modulator housing.

2. Tune the acoustic oscillator throughout the useful range (from about 25-160 MHz), and observe the diffracted spots on either side of the undiffracted beam. At 5 different frequencies (25, 50, 80, 110, 150 MHz), measure the angle of diffraction of all the diffracted spots by ruler. Do this using the red He-Ne. Confirm the angle measured with theory. Remember to take into account the index of refraction of the modulator.

3. Repeat part 1 and 2 using the Green HeNe laser (543 nm). Do you observe the optical and acoustical frequency dependence predicted in part 1 and 2 of the pre-Lab?

4. Repeat part 1 and 2 at 80 MHz only using the Orange-Yellow He-Ne. From the angles measured, compute the wavelengths of the orange and yellow line emitted by this laser.

5. Measure the diffraction efficiency as a function of acoustic power. Use the red HeNe laser and set the acoustic oscillator at 80 MHz. First, lineup the laser by repeating part 1. With the oscillator turned off, measure the transmitted laser power using a power meter. Then, measure the power in the undiffracted spot and in each diffracted spot as a function of the acoustic driving voltage. (Assume that the impedance of the modulator is constant, the power delivered to the modulator is then linearly proportional to the applied voltage.) Use 10 different voltages evenly spaced from zero to the max. Note that the power of the 1st order spot increases linearly with the drive voltage while the power of the 2nd order spot increases proportionally to the square of the drive voltage. Graph and compare to Eq. 21. Can you explain this behavior using the equations given in the background?
**Additional Information on Equipment:**

**NEC OD-8811 A.O. Modulator:** The modulator is made out of Lead Molybdate ($PbMoO_4$) with the following device parameters

<table>
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<th>$\rho$ ($g/cm^2$)</th>
<th>$v_s$ (km/sec)</th>
<th>$n$ (ref. index)</th>
<th>$p$ (photoelastic)</th>
<th>$M_2$</th>
<th>$L$ (mm)</th>
</tr>
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<tr>
<td>6.95</td>
<td>3.66</td>
<td>2.30</td>
<td>0.28</td>
<td>0.22</td>
<td>5</td>
</tr>
</tbody>
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(a) \[ k_d \]

(b) \[ k = k_d + k_s \]

(c) \[ k = k_d - k_s \]

(d) \[ \Lambda \]

\[ \begin{align*}
\theta &= \theta_s \\
\omega &= \omega_s
\end{align*} \]
Bragg Diffraction: Experimental Procedure

**Note**: Be **VERY** gentle in handling all HeNe laser tubes. The glass tube inside can be very easily **destroyed** from shock force. **DO NOT** drop the tubes or pile them one on top of the other. **DO NOT** tighten the holding screw with more than gentle finger force.

1. Set the angle of incidence to be 0° by retro-reflecting the beam off of the modulator back into the laser, or rotating the modulator until the diffraction pattern is symmetric. Use the green HeNe laser ($\lambda$=541 nm) for this part. Be sure that the laser beam is not “clipped” by the aperture of the modulator housing. Begin with the broadband modulator (the one with the larger base).

2. Tune the acoustic oscillator through the useful range (from about 20 MHz to about 140 MHz) by varying the drive frequency and observing the diffracted spots on either side of the fundamental beam. Measure the position of the diffracted spots relative to the fundamental spot with a ruler for drive frequencies of 20, 40, 60, 80, 100, 120, and 140 MHz. Also measure the distance from the exit face of the modulator to the plane where the position of the diffracted spots is measured. Calculate the diffraction angle as a function of drive frequency. Compare the measured angle values to those based on the theory. Remember to take into account the index of refraction of the modulator.

3. Repeat Steps 1 and 2 above replacing the green HeNe with a red HeNe ($\lambda$=632.8 nm). How does the optical and acoustic frequency dependence predicted in questions 1 and 2 of the prelab compare with your measurements?

4. Change to the second, high efficiency modulator (the one with the smaller base) and re-align.

5. Measure the diffraction efficiency as a function of acoustic power. Set the drive frequency to 80 MHz. First, line up the laser by repeating Step 1 above. With the oscillator turned off, measure the transmitted laser power by using a photodetector and a current meter. Then, measure the power in the non-diffracted spot, and in each diffracted spot as a function of the acoustic drive voltage. If one assumes that the impedance of the modulator is constant, then the power delivered to the modulator is linearly proportional to the applied voltage. Use 10 different voltages spaced evenly from zero to the max. Note that the power of the first-order spot increases linearly with the drive voltage, while the power of the second-order spot increases proportionally to the square of the drive voltage. Graph and compare your results with the predicted values obtained using Eq. 21. Explain this behavior using the equations given in the handout for this experiment.