5.2.2. Sketching Bode Plots. Bode plots are, in general, simpler to sketch by using straight-line asymptotic approximations. These will now be discussed.

Case 1.

(i) \( G(i\omega) = K \). We have — for \( K > 0 \):

\[
G_{dB} = 20 \log_{10} K, \quad \text{and} \quad \angle G(i\omega) = 0^\circ \quad (\text{or} \quad 2n\pi).
\]

(ii) \( G(i\omega) = i\omega \):

\[
G_{dB} = 20 \log_{10} \omega, \quad \omega > 0, \quad \text{and} \quad \angle G(i\omega) = 90^\circ.
\]

Since \( G_{dB} \) is considered as a function of \( \log_{10} \omega \), it is, in this case, a straight line with slope \( 20 \text{dB/decade} \) — on the Bode Gain plot.

Note that

\[
G(i\omega) = \frac{1}{i\omega} \Rightarrow G_{dB} = -20 \log_{10} \omega, \quad \omega > 0, \quad \text{and} \quad \angle G(i\omega) = -90^\circ.
\]

Therefore the Bode Plots in this case are simply the “negative” symmetries of those of \( G(i\omega) = i\omega \).

In general, for \( m \in \mathbb{Z}, \omega > 0 \),

\[
G(i\omega) = K [i\omega]^m \Rightarrow G_{dB} = 20 \log_{10} K + 20m \log_{10} \omega, \quad \angle G(i\omega) = 90^\circ \times m.
\]

Case 2. First Order Systems.

\[
G(i\omega) = \frac{a}{a + i\omega} = \frac{1}{1 + i\omega}.
\]

Therefore,

\[
G_{dB} = -20 \log_{10} |1 + i\omega|.
\]

To proceed we define

\[
G^\ell_{dB} := \text{Approx of } G_{dB} \text{ for } \omega \ll a, \quad \text{i.e., for low (\ell) frequencies,}
\]

\[
\approx -20 \log_{10} 1 = 0.
\]

We therefore conclude that \( G_{dB} \to 0 \) asymptotically for low frequencies. Therefore in this case

\[
\angle G^\ell_{dB}(i\omega) = 0^\circ.
\]

This can also be seen by noting that

\[
\angle G(i\omega) = -\angle(1 + i\omega) = -\tan^{-1} \left( \frac{\omega}{a} \right) \approx 0, \quad \text{for} \quad \omega \ll a.
\]

Similarly, define

\[
G^h_{dB} := \text{Approx of } G_{dB} \text{ for high (h) frequencies, i.e., for } \omega \gg a,
\]

\[
\approx -20 \log_{10} \frac{\omega}{a} = -20 \log_{10} \omega + 20 \log_{10} a.
\]
which is a line with a slope of -20 dB/decade, and
\[
\angle G_{dB}^h(i\omega) = -90^0,
\]
since
\[
-\tan^{-1}\left(\frac{\omega}{a}\right) \approx -90^0, \quad \text{for } \omega \gg a.
\]
Note that at
\[
\omega = a \quad \Rightarrow \quad G_{dB}^h = -20 \log_{10} \frac{a}{a} = -20 \log_{10} a = 0 = G_{dB}^f.
\]
Hence the two asymptotes intersect at a. Moreover at
\[
\omega = a \quad \Rightarrow \quad G(ia) = \frac{1}{1+i}, \quad \text{and} \quad \angle G(ia) = -45^0,
\]
\[
\Rightarrow \quad G_{dB} = 20 \log_{10} 1 - 20 \log_{10} (|i + 1| = \sqrt{2}).
\]
Therefore,
\[
\omega = a \quad \Rightarrow \quad G_{dB} = -3.01 \text{ dB}.
\]
The point
\[
(\omega = a \ (\text{or} \ \log_{10} a), \ 3 \text{ dB})
\]
on the gain plot of \(G(i\omega)\) is called 3 dB point and \(\omega = a\) is defined as break (corner) frequency. Note that at break frequency, the error between \(G_{dB}\) and its asymptotic approximation is the biggest one!


Consider
\[
G(s) = \frac{K}{s^2 + 2\zeta \omega_n s + \omega_n^2}, \quad \text{with} \quad \zeta \in (0, 1),
\]
i.e., the poles are complex conjugates.

To proceed, let us write
\[
G(i\omega) = \frac{K}{\omega_n^2} \left(1 - \frac{\omega^2}{\omega_n^2}\right) + i2\zeta \frac{\omega}{\omega_n}.
\]
From which it follows that
\[
G_{dB} = G_{dB1} + G_{dB2},
\]
where
\[
G_{dB1} := 20 \log_{10} \frac{K}{\omega_n^2},
\]
and
\[
G_{dB2} := -20 \log_{10} \left[(1 - \frac{\omega^2}{\omega_n^2})^2 \left(2\zeta \frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}.
\]
The Bode Plots for $G_{dB1}$ are straightforward and can simply be added to those of $G_{dB2}$.

We have, for low frequencies: $\omega \ll \omega_n$,

$$G_{dB2}^\ell = -20 \log_{10} \left[ \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2 \right]^{\frac{1}{2}} \approx 0 \text{ dB}, \quad \angle G_{dB2}^\ell = 0^\circ,$$

and for high frequencies: $\omega \gg \omega_n$,

$$G_{dB2}^h = -20 \log_{10} \left[ \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2 \right]^{\frac{1}{2}} \approx -40 \log_{10} \frac{\omega}{\omega_n} \text{ dB}, \quad \angle G_{dB2}^h = 180^\circ.$$

Next, we have, at $\omega = \omega_n \Rightarrow G_{dB}^h = 0$,

that is $\omega_n$ is the frequency where the 2 asymptotes meet. Moreover, as $\omega$ increases from $\omega_n$ to $10\omega_n$, i.e., through a decade, $G_{dB}^h$ becomes -40 dB. Therefore, slope of the high frequency asymptote is -40 dB/decade.

**Remark 5.6.** Note that the Bode Plots of a cascade system can be obtained by “adding” the Bode Plots of the individual systems.

### 5.2.3. Recap on Bode Plots (BP).

Begin with a FRF

$$G(i\omega) = |G(i\omega)| \angle G(i\omega), \quad \omega \in \mathbb{R}.$$

- **Gain** $|G(i\omega)|$.

  On a BP $|G(i\omega)|$ is “converted” into $G_{dB}$ defined by

  $$G_{dB} := 20 \log_{10} |G(i\omega)|.$$

  A Bode Gain Plot is a plot of $G_{dB}$ — expressed in deciBel, i.e., vertical distance represents dB — versus frequency $\omega > 0$. However, scale of the horizontal $\omega$-axis is chosen to be logarithmic. Therefore the horizontal $\omega$-axis is actually log$_{10}$ $\omega$-axis.

  On the vertical axis, a ratio (or factor) of 10 : 1 is 20 dB, ratio 2 : 1 is 6 dB.

  On the horizontal log$_{10}$ $\omega$-axis distance represents a frequency ratio, ratio 10 : 1 is called a decade, ratio 2 : 1 is an octave.

  Note that slope of a straight line on a Bode Gain Plot is described in units dB/decade or dB/octave.

- **Phase Shift** $\angle G(i\omega)$.

  A Bode Phase Shift plot is a plot of $\angle G(i\omega)$, in degrees, versus the same horizontal log$_{10}$ $\omega$-axis.

We now consider some applications of BP.