Example 4.6.

\[ Q(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15. \]

We find

\[
\begin{array}{ccc}
  s^5 & 1 & 3 & 10 \\
  s^4 & 2 & 6 & 15 \\
  s^3 & 0 & 2.5 & . \\
  s^2 & . & . & . \\
  s^1 & . & . & . \\
  s^0 & . & . & . \\
\end{array}
\]

Let us replace 0 by \( \epsilon \) and proceed. We find

\[
\begin{array}{ccc}
  s^5 & 1 & 3 & 10 \\
  s^4 & 2 & 6 & 15 \\
  s^3 & \epsilon & 2.5 & . \\
  s^2 & 6 - \frac{5}{\epsilon} & 15 & . \\
  s^1 & \star & . & . \\
  s^0 & 15 & . & . \\
\end{array}
\]

where

\[ \star := \frac{30\epsilon - 25 - 30\epsilon^2}{12\epsilon - 10}. \]

Letting \( \epsilon \to 0 \) but \( \epsilon \neq 0 \), we obtain

\[
\begin{array}{ccc}
  s^5 & 1 & 3 & 10 \\
  s^4 & 2 & 6 & 15 \\
  s^3 & \epsilon & 2.5 & . \\
  s^2 & -\frac{5}{\epsilon} & 15 & . \\
  s^1 & 2.5 & . & . \\
  s^0 & 15 & . & . \\
\end{array}
\]

where

\[ 6 - \frac{5}{\epsilon} \to -\frac{5}{\epsilon}, \quad \text{as} \quad \epsilon \to 0, \]

and

\[ \star := \frac{30\epsilon - 25 - 30\epsilon^2}{12\epsilon - 10} = \frac{3\epsilon}{1.2\epsilon - 1} + \frac{-2.5}{1.2\epsilon - 1} + \frac{-3\epsilon^2}{1.2\epsilon - 1} \to 2.5, \quad \epsilon \to 0. \]

Therefore \( Q(s) \) has 2 roots in the RHP.

Remark 4.7. Alternately, one can define

\[ Q_{new}(s) := Q\left(\frac{1}{s}\right). \]

Then proceed with forming the RA for \( Q_{new}(s) \).
Example 4.8.

\[ Q(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15. \]

Alternate Method: Form

\[ Q\left(\frac{1}{s}\right) = \frac{1}{s^5} + +2\frac{1}{s^4} + 3\frac{1}{s^3} + 6\frac{1}{s^2} + 10\frac{1}{s} + 15. \]

From which it follows that

\[ s^5 Q\left(\frac{1}{s}\right) = 15s^5 + 10s^4 + 6s^3 + 3s^2 + 2s + 1 := Q_{\text{new}}(s). \]

The RA for \( Q_{\text{new}}(s) \) also has 2 sign changes as expected:

\[
\begin{array}{cccc}
  s^5 & 15 & 6 & 2 \\
  s^4 & 10 & 3 & 1 \\
  s^3 & \frac{15}{10} & \frac{1}{2} & . . . \\
  s^2 & -\frac{1}{3} & 1 & . . . \\
  s^1 & \frac{75}{4} & . . . \\
  s^0 & 1 & . . . \\
\end{array}
\]

RA with a Zero Row (ZR). If RA cannot be completed due to a zero row, that is, a row with all zero entries, then one can proceed as follows:

(i) Form an auxiliary polynomial \( Q_{\text{aux}}(s) \) — from entries of the row immediately above the ZR — \( Q_{\text{aux}}(s) \) has only alternate powers of \( s \). Its highest power is the power shown in the leftmost column of the row just above the ZR.

(ii) Then, either

   a) Factor \( Q(s) \) as

   \[ Q(s) = Q_{\text{aux}}(s).\hat{Q}(s). \]

   Then we have

   \[
   \text{Roots of } Q(s) \in \text{RHP} = \text{Roots of } Q_{\text{aux}}(s) \in \text{RHP} + \text{Roots of } \hat{Q}(s) \in \text{RHP},
   \]

   or

   b) Let the ZR be the \( s^i \)-row, then its zeroes can be replaced with the coefficients of the polynomial

   \[ \frac{d}{ds} Q_{\text{aux}}(s). \]

Example 4.9.

\[ Q(s) = s^5 + 6s^4 + 15s^3 + 30s^2 + 44s + 24. \]
The RA is in this case:

\[
\begin{array}{cccc}
  s^5 & 1 & 15 & 44 \\
  s^4 & 6 & 30 & 24 \\
  s^3 & 10 & 40 & . \\
  s^2 & 6 & 24 & . \\
  s^1 & 2 & 0 & 0 \\
  s^0 & 24 & 0 & 0 \\
\end{array}
\]

\[\text{← (original) } ZR\]

Method a) We form

\[\tilde{Q}_{aux}(s) := 6s^2 + 24 = 6(s^2 + 4).\]

Therefore we can take

\[Q_{aux}(s) := s^2 + 4 \implies \text{roots: } i2, -i2.\]

Consequently

\[\hat{Q}(s) := \frac{Q(s)}{Q_{aux}(s)} = \frac{Q(s)}{s^2 + 4} = s^3 + 6s^2 + 11s + 6.\]

The RA for \(\hat{Q}(s)\) is:

\[
\begin{array}{cccc}
  s^3 & 1 & 11 \\
  s^2 & 6 & 6 \\
  s^1 & 10 & . \\
  s^0 & 6 & . \\
\end{array}
\]

Therefore, there exist no sign changes, that is, \(\hat{Q}(s)\) has no roots in RHP. As a consequence \(Q(s)\) has 2 imaginary roots (of \(Q_{aux}(s)\)).

Method b) We have

\[
\frac{d}{ds} Q_{aux}(s) = \frac{d}{ds} s^2 + 4 = 2s.\]

The RA now becomes

\[
\begin{array}{cccc}
  s^5 & 1 & 15 & 44 \\
  s^4 & 6 & 30 & 24 \\
  s^3 & 10 & 40 & . \\
  s^2 & 6 & 24 & . \\
  s^1 & 2 & 0 & 0 \\
  s^0 & 24 & 0 & 0 \\
\end{array}
\]

\[\text{← (original) } ZR\]

as expected! However, since \(Q_{aux}(s)\) is a factor of \(Q(s)\) therefore, as before,

\[Q(s) = 0 \implies Q_{aux}(s) = s^2 + 4 = 0 \implies \text{roots } \pm i2.\]
Finally note that
(i) If

\[ Q(s) \text{ has negative coefficients } \Rightarrow \text{ roots } \in \text{ RHP}. \]

(ii) Any of the coefficients \( a_n, n \neq 0 \), of \( Q(s) \) is missing \( \Rightarrow \) Roots in RHP, or on imaginary axis.