2.4. Stability, Transient and Steady-State Responses

Let $y(\cdot)$ be the output of a SISO, LTIC system due to an input $u(\cdot)$ then we know from 2.2.2 (ii) that $y(\cdot)$ admits the decomposition

$$y(\cdot) = y_{zi}(\cdot) + y_{zs}(\cdot).$$

If the zero-input ($zi$) response $y_{zi}(t)$ (i.e., response of the system due only to initial conditions since the input is taken to be zero) of a system goes to 0 as $t$ becomes large

$$y_{zi}(t) \rightarrow 0, \quad t \rightarrow \infty,$$

for all initial conditions (to be defined as initial states later on), then the system is defined to be asymptotically stable (AS).

It is evident that, since input is not involved in the response $y_{zi}(\cdot)$ — hence the subscript “zi” — “AS” must be a consequence of the locations of the poles of the system transfer function $T(s)$ in the complex s-plane. In other words, if all the poles of $T(s)$ “live” in the open LHP (i.e., the LHP without the imaginary axis) then the system is AS. Conversely, if a system is AS then all the poles of $T(s)$ are in the open LHP.

A system is called Bounded Input Bounded Output (BIBO) stable if the zero-state (zs) (or zero initial conditions) response $y_{zs}(\cdot)$ (i.e., the response due only to the input) is bounded whenever the corresponding input $u(\cdot)$ is bounded, i.e.,

$$(2.10) \quad \forall \, t : |u(t)| < a \Rightarrow |y_{zs}(t)| < b, \quad \forall \, t,$$

where $a$ and $b$ are positive constants.

Let $h(\cdot)$ denote the impulse response function of a SISO, LTI system. Then AS is equivalent to

$$(2.11) \quad h(t) \rightarrow 0, \quad t \rightarrow \infty.$$ 

This, in turn, is equivalent to absolute integrability of $h(\cdot)$, i.e.,

$$(2.12) \quad \int_0^\infty |h(t)| \, dt < \infty.$$ 

It can be shown that (2.12) is equivalent to BIBO stability. Hence BIBO Stability and Asymptotic Stability are equivalent—provided $T(s)$ does not admit poles and zeros cancellation (PZC)!

What happens when there exists PZC? Suppose there are some poles in RHP, and suppose zeros of $T(s)$ “sit” on top of all these RHP poles. In this case the IRF goes to 0 as $t \rightarrow \infty$, while, at least one term in the zero-input response will go to $\infty$, as $t \rightarrow \infty$. 

b) Transient (tr) Response and Steady-State (ss) Response. Heuristically speaking, the steady-state response $y_{ss}(\cdot)$ of a LTI system is the output which “still” exists after a “long long time”, while the transient response $y_{tr}(\cdot)$ is the output which only exists for a short time! The existence of $y_{ss}(\cdot)$ implies that the system is “stable” in some suitable sense.

**Example 2.7. BBE**

We now define.

**Definition 2.8.** The steady-state response $y_{ss}(\cdot)$ of a LTI C and stable system is the output that exists for a long time—after an input is applied, that is,

$$y_{ss}(t) < \infty, \quad t \to \infty,$$

while its transient response $y_{tr}(\cdot)$ is the output which will eventually disappear, that is,

$$y_{tr}(t) \to 0, \quad t \to \infty.$$

It is evident that $y_{tr}(\cdot)$ is a consequence of the system stable poles while $y_{ss}(\cdot)$ is a consequence of the system input $u(\cdot)$. Output $y(\cdot)$ of the system can therefore be decomposed into

$$y(\cdot) = y_{ss}(\cdot) + y_{tr}(\cdot).$$

Note that this decomposition is not the same as that of (2.2) which always exists, while (2.13) only exists for a stable system.

**Example 2.9. BBE**

One can see the meaning of $y_{tr}(\cdot)$ and $y_{ss}(\cdot)$ for the case in which the system transfer function and the input Laplace transform are rational functions of $s$ as follows.

Let $T(s)$ be the transfer function of a LTIC system and suppose that it is a rational function of $s$ and is of the form

$$T(s) := \frac{n_T(s)}{d_T(s)}.$$

Let $U(s)$ be a rational function of $s$ and is an input transform of the form

$$U(s) = \frac{n_U(s)}{d_U(s)}.$$

The the corresponding output is then

$$Y(s) = T(s) \cdot U(s) = \frac{n_T(s) \cdot n_U(s)}{d_T(s) \cdot d_U(s)}.$$
This can be further written as — by a PFE,
\[ Y(s) = \frac{n_1(s)}{d_T(s)} + \frac{n_2(s)}{d_U(s)}. \]
It then follows that if the poles of \( T(s) \) are in the open LHP, i.e., if the system is stable, then the inverse Laplace Transform of the first term on the right hand side must be such that
\[ \mathcal{L}^{-1}\left\{ \frac{n_1(s)}{d_T(s)} \right\} \rightarrow 0, \quad t \rightarrow \infty. \]
Therefore, it is called the transient response of the system and is denoted by
\[ y_{tr}(t) := \mathcal{L}^{-1}\left\{ \frac{n_1(s)}{d_T(s)} \right\}. \]
Note that \( y_{tr}(t) \) depends only on the poles of \( T(s) \) — which are stable poles.

If the second term
\[ \frac{n_2(s)}{d_U(s)} \]
has poles on the imaginary axis as well as poles in the open LHP. Then the response \( \mathcal{L}^{-1}\left\{ \frac{n_2(s)}{d_U(s)} \right\} \) will not go to 0 as \( t \rightarrow \infty \). Therefore we define the steady-state response
\[ y_{ss}(t) := \mathcal{L}^{-1}\left\{ \frac{n_2(s)}{d_U(s)} \right\}. \]
We note that \( y_{ss}(t) \) is a direct consequence of the input \( u(t) \) while the poles of the systems have nothing to do with it!

### 2.5. Steady-State Errors

We now introduce the concept of Steady-State Error (SSE) which plays a central role in tracking problems.

We shall adopt the following definition of stability.

**Definition 2.10.** Let \( S \) be a LTIC system with Transfer Function \( S(s) \) whose poles are denoted by \( p_i, i = 1, 2, \ldots, N \). The system \( S \) is defined to be stable if \( \Re p_i < 0 \).

In the discussion of this Section the plant \( P \) is taken to be stable.

Consider the FCS of Figure 2.2 whose transfer function \( T_{cl}(s) \) is given by (2.6) and (2.7)
\[
T_{cl}(s) := \frac{Y_p(s)}{R(s)} = \frac{G_{co}(s)G_p(s)}{1 + G_{co}(s)G_p(s)H(s)} = \frac{G(s)}{1 + G(s)H(s)}.
\]
where
\[ G(s) := G_{co}(s) G_p(s). \]

Now, let us compute the transform \( E(s) \) of the error signal \( e(t) \) defined by
\[ (2.15) \quad e(t) := r(t) - y_p(t), \quad t \geq 0. \]
We have
\[ (2.16) \quad E(s) = R(s) - Y_p(s) = R(s) - T_{cl}(s) R(s) = [1 - T_{cl}(s)] R(s). \]

We make the following Definition.

**Definition 2.11.** The **Steady-State Error** (SSE) \( e_{ss-r(t)} \) — due to a given reference-signal \( r(\cdot) \) — of a non-unity feedback system is defined as
\[ (2.17) \quad e_{ss-r(t)} := \lim_{t \to \infty} e(t). \]

**Remark 2.12.** Let \( f(\cdot) \) be a function of the real variable \( t \). Recall that:

"\( f(t) \to a \) as \( t \to \infty \"

means:

If for each \( \epsilon > 0 \) there is a real number \( b > 0 \) such that:
\[ |f(t) - a| < \epsilon \quad \text{for all} \quad t \quad \text{satisfying} \quad |t| > b, \]
that is if \( f(t) \) is “closed” to \( a \) for \( t \) “big enough”. The number \( a \) is called the limit of \( f(t) \) and we write
\[ \lim_{t \to \infty} f(t) = a. \]

It follows from the Final-Value Theorem that
\[ (2.18) \quad e_{ss-r(t)} := \lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s). \]

Therefore by (2.16)
\[ (2.19) \quad e_{ss-r(t)} = \lim_{s \to 0} \left\{ s \left[ 1 - T_{cl}(s) \right] R(s) \right\}, \]
\[ (2.20) \quad = \lim_{s \to 0} \left\{ s \frac{1 + G(s) H(s) - G(s)}{1 + G(s) H(s)} \cdot R(s) \right\}. \]
This shows that \( e_{ss-r(t)} \) depends on both \( T_{cl}(s) \) and \( R(s) \).

Setting \( H(s) = 1 \) we obtain the **SSE** \( e_{ss1-r(t)} \) — for **unity-feedback**,
\[ (2.21) \quad e_{ss1-r(t)} = \lim_{s \to 0} \left\{ s \frac{1}{1 + G(s)} \cdot R(s) \right\}. \]
Remark 2.13. Let \( F(s) \) be the Laplace Transform of \( f(t) \) then, by The Final-Value Theorem,
\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s),
\]
provided
\[
f(\infty) := \lim_{t \to \infty} f(t) < \infty,
\]
i.e., \( f(\infty) \) must exist, and \( f'(t) \) is piecewise-continuous for \( t \geq 0 \). The existence of \( f(\infty) \) is actually equivalent to requiring that the domain of convergence of \( sF(s) \) contains the point \( s = 0 \). However, the limit of \( sF(s) \) as \( s \to 0 \) may exist even though \( f(\infty) \) does not exist, for example, take \( f(t) = \sin t \),
\[
F(s) = \frac{1}{s^2 + 1} \quad \Rightarrow \quad sF(s) = \frac{s}{s^2 + 1} \to 0, \quad s \to 0.
\]
But \( \sin(\infty) \) does not exists! Therefore we cannot set \( \sin(\infty) = 0! \)

We now compute SSE for various “test” reference-inputs \( r(\cdot) \).

**2.5.1. SSE:** \( e_{ss-1(t)} \), \( e_{ss1-1(t)} \), and \( R(s) = \frac{1}{s} \). It is plain that the simplest reference-input signal is \( R(s) = \frac{1}{s} \) — i.e., \( r(t) = 1(t) \), the unit step function (written as \( 1(t) \)). In this case the SSE \( e_{ss-1(t)} \) due to \( 1(t) \) is
\[
e_{ss-1(t)} = \lim_{s \to 0} \{ s \left[ 1 - T_{cl}(s) \right] \frac{1}{s} \} = 1 - T_{cl}(0).
\]
and, setting \( H(s) = 1 \), i.e., for unity-feedback,
\[
e_{ss1-1(t)} = \lim_{s \to 0} \left\{ s \cdot \frac{1}{1 + G(s)} \cdot \frac{1}{s} \right\} = \frac{1}{1 + G(0)}.
\]
It follows from (2.23) that
\[
e_{ss1-1(t)} = 0 \iff G(0) = \infty \iff \exists \text{ pole}(s) = 0.
\]

**Remark 2.14.** (i) Recall that (see subsection 2.2.2 (i)) a system whose transfer function \( G(s) \) does not have a pole at 0 is called a type 0 system. Similarly, a system whose transfer function \( G(s) \) has a pole \( p = 0 \) (respectively, \( p^{\ell} = 0 \)) of order 1 (respectively, of order \( \ell \geq 1 \)) is called a type 1 (respectively, type \( \ell \)) system, etc.

Therefore, if \( G(s) \) is:

\[
\begin{align*}
\text{Type 0} & : \quad e_{ss1-1(t)} = \frac{1}{1 + G(0)} \neq 0, \\
\text{Type } \geq 1 & : \quad e_{ss1-1(t)} = 0.
\end{align*}
\]