Remarks on Controllability and Observability

Given a pair \((A, B)\):

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) \text{ given.} \tag{1}
\]

We define the uncontrollable subspace

\[
M_{uc} := \{x \in \mathbb{X} : B^* A^* k x = 0_X, \, k = 0, 1, 2, ..., n - 1\}, \tag{2}
\]

where \(A\) is \(n \times n\), and the controllable subspace

\[
M_c := M_{uc}^\perp. \tag{3}
\]

If \(M_{uc} \neq \{0_X\}\) then the system, or the pair \((A, B)\), or the state space \(X\), is uncontrollable. On page 25 of “BLN” we mention that in this case: “only states belong to \(M_c\) are controllable”. This implies that not all uncontrollable states are living in \(M_{uc}\). In fact any state not belonging to \(M_c\) is uncontrollable.

Similarly, if \(M_{uo} \neq \{0_X\}\) then any state not living in \(M_o := M_{uo}^\perp\) is unobservable.

In HW2-#5 you found

\[
M_c := \operatorname{span}\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\}, \tag{4}
\]

\[
M_{uc} := \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\}, \tag{5}
\]

\[
M_o := \operatorname{span}\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}, \tag{6}
\]

and

\[
M_{uo} := \operatorname{span}\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}. \tag{7}
\]
Now let us look for a state \( x_{c-u0} \) which is controllable and uncontrollable at the same time. For this we can first consider \( M_c \cap M_{u0} \). A state \( x_c \in M_c \) is of the form

\[
x_c = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\beta \\ \alpha \\ -\beta \end{bmatrix}
\] (8)

Now to make \( x_c \) to live in \( M_{u0} \) at the same time, we must require that

\[
\begin{bmatrix} 2\beta \\ \alpha \\ -\beta \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma \\ \gamma \end{bmatrix}.
\] (9)

Therefore

\[
\alpha = \beta = \gamma = 0.
\] (10)

Consequently

\[
M_c \cap M_{u0} = \{0_X\}.
\] (11)

Note: This can be easily seen by simply “eyeballing” (sorry I cannot think of a better term!)

Now, in view of our remarks above, we can look for a state \( x_c \) in \( M_c \) which is Not living in \( M_o \), i.e., \( x_c \) is not observable, that is it can be identified as \( x_{c-u0} \). One such state is clearly

\[
x_{c-u0} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

The aim of this not is to clarify possible “confusion” which may exist.