3.3. Steady-State Errors

Consider the FCS of Figure 3.2 whose transfer function $T_{cl}(s)$ is given by (3.5) and (3.6)

$$T_{cl}(s) := \frac{Y_p(s)}{R(s)} = \frac{G_{td}(s)G_p(s)}{1 + G_{td}(s)G_p(s)} \frac{H(s)}{1 + G(s)H(s)}$$

(3.14)

Now, let us compute the transform $E(s)$ of the error signal $e(t)$

$$e(t) := r(t) - y_p(t), \quad t \geq 0.$$  

(3.15)

We have

$$E(s) = R(s) - Y_p(s) = R(s) - T_{cl}(s)R(s) = [1 - T_{cl}(s)]R(s).$$

(3.16)

Let $e_{ss}$ be Steady-State Error, of a non-unity feedback system, defined by

$$e_{ss} := \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

— because of the Final-Value Theorem. Therefore by (3.16)

$$e_{ss} = \lim_{s \to 0} \left\{ s \left[ 1 - T_{cl}(s) \right] \frac{1}{s} \right\} = 1 - T_{cl}(0).$$

(3.17)

The next simple reference-input is $R(s) = \frac{1}{s^2}$, i.e., $r(t) = tU(t)$ — ramp reference-input,

$$e_{ss-ramp} = \lim_{s \to 0} \left\{ s \left[ 1 - T_{cl}(s) \right] \frac{1}{s^2} \right\} = \lim_{s \to 0} \frac{1 - T_{cl}(0)}{s}.$$  

(3.18)

Therefore, if

(i) $T_{cl}(0) \neq 1$, then $e_{ss-ramp}$ is infinite.
(ii) $T_{cl}(0) = 1$, then

$$e_{ss-ramp} = 0 \Rightarrow e_{ss-ramp} = \lim_{s \to 0} \left[ -\frac{dT_{cl}(s)}{ds} \right].$$

This can be rewritten as

$$e_{ss-ramp} = \lim_{s \to 0} \left[ -\frac{1}{T(s)} \frac{dT_{cl}(s)}{ds} \right] = \lim_{s \to 0} \left[ -\frac{d\ln T_{cl}(s)}{ds} \right],$$

(3.21)

where use has been made of the fact that $T(0) = 1$.

It then follows that if $T_{cl}(s)$ is of the form

$$T_{cl}(s) := K \frac{(s - z_1)(s - z_2)...(s - z_m)}{(z - p_1)(s - p_2)...(z - p_n)},$$

(3.22)
where $K$ is a constant — called the **gain** of $T_{cl}(s)$, $z$ stands for zero and $p$ stands for pole. Then it is easy to see that

\[ e_{ss-ramp} = \sum_{i=1}^{m} \frac{1}{z_i} - \sum_{i=1}^{n} \frac{1}{p_i}. \]  

This implies that $e_{ss-ramp}$ depends on zeros and poles of $T_{cl}(s)$ — but not on its gain $K$.

**Remark 3.4.** It follows from (3.16) and (3.14) that $e_{ss}$ can also be expressed in terms of $G(s)$ and $H(s)$ as

\[ e_{ss} = \lim_{s \to 0} s \left[ \frac{1 + G(s)H(s) - G(s)}{1 + G(s)H(s)} \right] R(s). \]  

We now turn to $e_{ss-uni}$ of a unity feedback system. We have from (3.24) with $H(s) = 1$

\[ e_{ss-uni} = \lim_{s \to 0} s \left[ \frac{1}{1 + G(s)} \right] R(s). \]  

Therefore

\[ e_{ss-uni-1step} = \lim_{s \to 0} s \left[ \frac{1}{1 + G(s)} \right] \frac{1}{s} = \lim_{s \to 0} \frac{1}{1 + G(s)}. \]  

This can be written as

\[ e_{ss-uni-1step} = \frac{1}{1 + K_p}, \quad K_p := \lim_{s \to 0} G(s) \] — called position error constant.

It then follows that if $G(s)$ has no zero-pole then $K_p \neq \infty$. Hence $e_{ss-uni-1step} \neq 0$. However, if $G(s)$ admits one or more zero-poles then $e_{ss-uni-1step} = 0$ since $K_p = \infty$.

Similarly, we find

\[ e_{ss-uni-ramp} = \lim_{s \to 0} s \left[ \frac{1}{1 + G(s)} \right] \frac{1}{s^2}. \]  

Therefore

\[ e_{ss-uni-ramp} = \lim_{s \to 0} \left[ \frac{1}{sG(s)} \right] = \frac{1}{K_v}, \]  

where

\[ K_v := \lim_{s \to 0} sG(s) \] — called velocity error constant.

In this case if $G(s)$ has no zero-poles then

\[ K_v = 0 \quad \Rightarrow \quad e_{ss-uni-ramp} = \infty. \]
However if
\[ G(s) \text{ has two or more zero poles } \Rightarrow e_{ss-uni-ramp} = 0. \]

**Problem.** Find \( e_{ss-uni-para} \) where “para” means the parabolic reference input \( \frac{1}{2}t^2U(t) \). There will be a constant which is called “acceleration error constant” \( K_a \).

**Remark 3.5.** One can define Transient Error \( e_{tr}(t) \) as
\[ (3.31) \quad e_{tr}(t) := e(t) - e_{ss}, \quad t \geq 0. \]