An Introduction To State-Feedback And Output-Feedback Controls

Nhan Levan

Author address:

Department of Electrical Engineering, University of California in Los Angeles, California, USA
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Preface

These notes are intended as an Introduction to Feedback Control Systems. The prerequisite is a course on Linear Systems — with Input-Output Description — covering both time-domain and s-domain analyses.

The notes are self-contained and are divided into two parts. Part 1 (Chapter 2) covers State-Feedback Control, while Part 2 (Chapter 3) deals with Output-Feedback Control — for Single-Input Single-Output Systems.

Nhan Levan
CHAPTER 1

Review and Preview

1.1. Time-Domain Analysis: Impulse Response Function (IRF)

In the following we will be dealing with Linear (L), Time-Invariant (TI), and Causal (C) systems, simply referred to as LTIC. In Chapter 2 we deal with Multiple-Input Multiple-Output (MIMO) systems, while Chapter 3 will only be involved with Single-Input Single-Output (SISO) systems.

A system $S$ is said to have an IOD (Input-Output Description) if it can be “represented” by an input-output relation of the form

$$y(\cdot) = T[u(\cdot)],$$

where $T[\cdot]$ is a transformation taking inputs $u(\cdot)$ to outputs $y(\cdot)$. Moreover, each output $y(\cdot)$ is uniquely determined by an input $u(\cdot)$. A system $S$ is called L if $T[\cdot]$ is a linear transformation, i.e., it satisfies the following property

$$T[\alpha u_1 + \beta u_2] = \alpha T[u_1] + \beta T[u_2],$$

for any scalars $\alpha, \beta$, and any inputs $u_1, u_2$ — belonging to a vector space $U$ — called input space.

A system $S$ is TI if

$$y(\cdot) = T[u(\cdot)] \Rightarrow T[u(\cdot) - a] = y(\cdot - a),$$

for any scalar $a$. This simply means that if an input is “shifted” (along the time axis) by an amount $a$, then the corresponding output is shifted by the same amount.

The Impulse Response Function (IRF) $h(t, \tau)$ of $S$ is the system output at time $t$ due to an “impulse” input $\delta(t-\tau)$ — applied at time $\tau$, i.e.,

$$h(t, \tau) := T[\delta(t - \tau)], \quad \forall \ t, \tau \in \mathbb{R}.$$ 

If $S$ is also causal then its IRF is of the form: $h(t, \tau)U(t - \tau)$. If $S$ is TI, in addition, then its IRF is of the form: $h(t - \tau)U(t - \tau)$.

The following results show that a L system can be completely “analysed” once its IRF is known.
Theorem 1.1. Let $h(t, \tau)U(t - \tau)$ be IRF of a LC system $S$, then
$S$ admits the IOD:

$$y(t) = \int_{-\infty}^{\infty} h(t, \tau)U(t - \tau)u(\tau) d\tau, \quad t \in \mathbb{R},$$

$$= \int_{-\infty}^{t} h(t, \tau)U(t - \tau)u(\tau) d\tau.$$

If $u(t) = 0$, $t < 0$, i.e., the input is of the form $u(t)U(t)$, and if the
system is also TI, then

$$(1.5) \quad y(t) = \int_{0}^{t} h(t - \tau)u(\tau) d\tau, \quad t \geq 0,$$

called convolution integral.

1.2. $s$-Domain ($L_s\{\cdot\}$) Analysis: Transfer Function (TF)

Let $f(t)$ be defined for $t \geq 0$ and is equal to 0 for $t < 0$. Then the
Laplace Transform $L_s\{\cdot\}$ of $f(t)$ defined as

$$(1.6) \quad L_s\{f(t)\} := \int_{0}^{\infty} e^{-st}f(t) dt := F(s),$$

provided the integral exists.

Let $h(t)U(t)$ be the IRF of a LTIC system $S$. Then

$$(1.7) \quad L_s\{h(t)U(t)\} := H(s)$$

is called System Function (SF) or Transfer Function (TF) of $S$. It
follows from taking $L_s$ of (1.5) that

$$(1.8) \quad Y(s) = H(s)U(s),$$

where

$Y(s) := L_s\{y(t)\}$ and $U(s) := L_s\{u(t)\}$.

Consequently, an alternate definition of TF is

$$(1.9) \quad H(s) = \frac{Y(s)}{U(s)}.$$

Equivalently,

$$(1.10) \quad H(s) := \frac{\text{Output due to } e^{st}, \forall t \in \mathbb{R}}{e^{st}, \forall t \in \mathbb{R}}.$$

The following results play an important role in Chapter 3.
**Theorem 1.2.** (i) Initial-Value Theorem: Let $F(s)$ be the Laplace Transform of $f(t)$. If the derivative $f'(t)$ is piecewise-continuous for $t \geq 0$, then

$$f(0^+) = \lim_{s \to \infty} sF(s).$$

(ii) Final-Value Theorem: If $\lim_{t \to \infty} f(t)$ exists, and $f'(t)$ is piecewise-continuous for $t \geq 0$, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s).$$

It is plain from this Theorem that the initial-value $f(0^+)$ and final-value $f(\infty)$ can be found from the values of $sF(s)$ as $s \to \infty$ and $s \to 0$, respectively. In other words, we can compute $f(0^+)$ and $f(\infty)$ without knowing $f(t)$, but knowing $F(s)$ instead.

**1.3. Transient and Steady-State Responses**

Let $S$ be a LTIC system with IRF $h(t - \tau)U(t - \tau)$. Let $y(\cdot)$ be the output of $S$ due to an input $u(\cdot)$. Then it can be shown that

$$y(\cdot) = y_{ss}(\cdot) + y_{tr}(\cdot),$$

where $y_{ss}(\cdot)$ is Steady State Response and $y_{tr}(\cdot)$ is Transient Response. Heuristically speaking, $y_{ss}(\cdot)$ is the output which “still” exists after a “long long time”, while the transient response $y_{tr}(\cdot)$ only exists for a short time! The existence of $y_{ss}(\cdot)$ implies that the system is, somehow, “stable” in some suitable sense.

**1.4. Preview**

These notes introduce you to Fundamentals of Feedback Controls. What you have done up to now is to analyse systems — in time-domain or in s-domain, but you have not “controlled” anything! In other words, you have not involved with control systems! In the following you will be introduced to control systems, in particular, “Feedback Control Systems” (FCS).

**1.4.1. Open-Loop (OL) and Closed-Loop (CL) Systems.** A LTI system $S$—often called a PLANT— is referred to as an OL system. If someone gives you such a system and if you do not do anything to it, all you can get out of the system are outputs when you apply inputs to it. If you can to do something to it—other than watching its outputs—then you can (proudly) broadcast to the wide world that you actually deal with a “control problem”. Hence, once you have successfully controlled the system—by hooking equipments/systems to it, applying “desired” inputs to it, etc. — the whole setup becomes an OL control system. In particular, if there is “connection” between
the system output terminals and its input terminals — possibly via additional systems and equipments—then the resulting overall system is a CL Control System.

In general, a system is called a Control System if it is used to control “something”. A system is called OL if its input and output terminals are not connected — directly or indirectly. This implies that if one has a system which is OL then, of course, its function is somewhat limited, since any “control” input to the system will produce a “control” output which depends only on the input and the system — but nothing else! A system is called CL if its input and output terminals are connected directly or indirectly — that is via other systems or equipments. This implies that its input, in some sense, are “affected” by its output!

In Chapter 2 we discuss **State-Feedback** control. Here “state” of a system is fed back to its input in such a way that the resulting CL system can perform specific tasks. The tasks which we will study are: Pole-Placement (PP), Stabilizing an unstable (OL) system, State Estimation, and Linear Quadratic Control (LQR).

Chapter 3 covers **Output-Feedback** Controls for SISO systems. Here output of a SISO system is fed back — directly or indirectly — to its input resulting in a CL system which can include other components needed for the CL system to perform prescribed tasks.

### 1.5. Problems

Please review EE102 — except Fourier Series and Fourier Transforms.