1. Consider the system
\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0,
\]
\[
y(t) = CX(t),
\]
where
\[
A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

(i) Find the subspaces \(M_{uc}\) and \(M_c\). Hence, conclude that the system is NOT completely controllable.

(ii) Verify that the eigenvalues of \(A\) are \(\{\lambda_1 = -1, \lambda_2 = -2\}\) and find the corresponding eigenvectors \(\{\phi_1, \phi_2\}\) of \(A\) and show that only one of the two eigenvectors lies in \(M_c\).

(iii) We now want to implement a state-feedback control with the control-law:
\[
u = Kx \quad \text{where} \quad K = [K_1 \quad K_2].
\]

Derive the state-space description of the closed-loop system and identify the closed-loop system matrix.

(iv) Show that no matter what \(K_1, K_2\) we choose, \(\lambda_1 = -1\) is always an eigenvalue of the closed loop system matrix. Hence, conclude that the pole ‘-1’ of
the open loop cannot be modified by state-feedback. Explain why this happens using your answer in (ii).

(v) Now find $K_1, K_2$ such that the closed-loop system matrix has the eigenvalues $\{-1, -5\}$ and conclude that the solution is not unique.

2. In the system given in (1), let

$$
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
$$

(i) Show that the system is uncontrollable (no need to find $M_c$ or $M_{uc}$).

(ii) Introduce a state-feedback control of the form $u = Kx + r$ where $K = [K_1 \ K_2 \ K_3]$ and find the state-space description of the closed-loop. (Hint: for the closed loop, $r(\cdot)$ is the ‘input’.)

(iii) Show that no matter what $K_1, K_2, K_3$ we choose, the closed-loop system is always uncontrollable.

This problem illustrates the fact that if the open-loop system is uncontrollable to begin with, then it cannot be made controllable in the closed-loop by using state-feedback. More generally, state-feedback does not affect controllability of a system.

3. Consider a unity-feedback system with

$$
G_p(s) = \frac{1}{s+2}, \quad G_c(s) = \frac{K}{s}, \quad R(s) = \frac{1}{s}. \quad (2)
$$

(i) Write down the expression for $y(t)$.

(ii) Using MATLAB, plot the output for $K = 1, 2, 10$. In each case, calculate the closed-loop poles. Your code should look something like:

```matlab
>> t = linspace(0, Tfinal, N); % this sets up the time axis from 0 to 'Tfinal' with 'N' equally spaced steps. Choose appropriate values for 'Tfinal' and 'N'

>> y = (whatever function of 't'); % Note: if, e.g., y(t)=t^2 then in your code this should appear as "t.^2"

>> plot(t,y);
```

(iii) Define the error between $r(t)$ and $y(t)$ as $\epsilon(t) := r(t) - y(t)$ and find its steady-state value $\epsilon_{ss}$.
4. Consider a unity feedback system with forward-path transfer-function

\[ G(s) = \frac{K}{s(s - 3)}. \] (3)

Find the steady-state error for \( R(s) = \frac{1}{s} \)

5. Determine the step and ramp error constants for the unity-feedback systems with the following forward-path transfer function. Also write down their steady-state errors for unit-step and unit-ramp inputs.

(i) \( G(s) = \frac{100}{s^2(s^2 + 10s + 100)} \)

(ii) \( G(s) = \frac{1000}{s(s + 10)(s + 100)} \)