

2. Concept: Partition.

3. Example. 4 red balls, 5 black balls. If we pick the first ball and put it aside, and then we pick the second ball. What’s the probability for the 1st ball to be red and the 2nd to be black? What’s the probability for the second picked ball to be red?

Let A denote the event that the first ball is red, and B be the event that the second ball is black.

\[ P[A \cap B] = P[B|A]P[A] = \frac{5}{8} \times \frac{4}{9}. \]

Let C denote the event that the 2nd picked ball is red, \(B_1\) be the event that the 1st ball is red, while \(B_2\) be the event that the 1st ball is black.

\[ P[C] = P[C \cap (B_1 \cup B_2)] = P[C|B_1]P[B_1] + P[C|B_2]P[B_2] = \frac{3}{8} \times \frac{4}{9} + \frac{4}{8} \times \frac{5}{9} \]

4. Example. We have a batch of contaminated bulbs, of which 90% are good, 10% are bad. If the bulb is good, \(P[LT > t] = e^{-\alpha t}\); however, if the bulb is bad, \(P[LT > t] = e^{-100\alpha t}\).

(a) What’s the probability for a randomly picked bulb’s LT to be larger than \(t\)?

Let A be the event that the LT of a randomly picked bulb is larger than \(t\), and \(B_1\) be the event that the picked bulb is good, \(B_2\) be the event that the picked bulb is bad.

\[ P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] = e^{-\alpha t} \cdot 0.9 + e^{-100\alpha t} \cdot 0.1. \]

Suppose a bulb will be shipped if it’s still working at time \(t = \frac{1}{10\alpha}\). Let A be the event that a bulb is shipped,

\[ P[A] = e^{-0.1} \cdot 0.9 + e^{-10} \cdot 0.1 = 0.8147. \]
(b) What’s the probability for a bulb to be good if it’s shipped?

Let A be the event that a bulb is shipped, we know $P[A] = 0.8147$.

$$P[B_1|A] = \frac{P[A \cap B_1]}{P[A]} = \frac{P[B_1|A|B_1]}{P[A]} = \frac{0.9 \times e^{-0.1}}{0.8147} = 0.999994.$$ 

Suppose the test time is $\frac{1}{1000}$, then

$$P[A|B_1] = e^{-0.01}

P[A|B_2] = e^{-1}

P[B_1|A] = 0.96.$$ 

(c) Does the exponential distribution have memory?

$$P[LT \geq T] = e^{-\alpha T}$$ 

$$P[LT \geq T + t_0|LT \geq t_0] = \frac{P[LT \geq T + t_0 \cap LT \geq T]}{P[LT \geq t_0]} = \frac{P[LT \geq T + t_0]}{P[LT \geq t_0]} = e^{-\alpha T}.$$ 

So the exponential distribution is memoryless.


6. Example. Suppose there’re two coins. The probability for coin 1 to be head is $p_1$, and the probability for coin 2 to be head is $p_2$. Then what’s the probability to get a head if we randomly flip a coin? What’s the probability for the coin to be coin 1 if we observe a head?

Let A be the event that we get a head by flipping a random coin, $B_1$ be the event that coin 1 being picked, and $B_2$ be the event that coin 2 being picked.

$$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] = \frac{p_1 + p_2}{2}.$$ 

$$P[B_1|A] = \frac{p_1}{p_1 + p_2}.$$ 


If we toss a coin 2 times, are the results independent?