1. Problem 2.42 We assume (reasonably) that each time we try to catch an animal this is as likely to be caught as any of the remaining uncaught animals. The number of ways we can capture 5 tagged animals out of the 10 animals previously tagged is \( \binom{10}{5} \).

The other 20-5 = 15 animals are untagged (belong to the remaining \( N - 10 \) animals) and therefore can be captured in \( \binom{N-10}{20-5} \) ways. It follows that the probability that 5 animals out of 20 are found to be tagged if we have a population of \( N \) animals of which 10 were previously tagged is:

\[
\frac{\binom{10}{5} \binom{N-10}{20-5}}{\binom{N}{20}} = P_5(N)
\]

Since \( P_5(N) \) represents the probability of the observed event when there are actually \( N \) animals present in the region, it would appear that a reasonable estimate of \( N \) would be the value of \( N \) that maximizes \( P_5(N) \). Such an estimate is called a maximum-likelihood estimate.

The maximization of \( P_5(N) \) can most simply be done by first noting that \( \frac{P_5(N)}{P_5(N-1)} = \frac{(N-10)(N-20)}{N(N-10-20+5)} \). Now the above ratio is greater than 1 if and only if \((N - 10)(N - 20) \geq N(N - 10 - 20 + 5)\) or, equivalently, if and only if \( N \leq 40 \). Thus, as \( N \) increases, \( P_5(N) \) first increases, and then decreases, and reaches its maximum value at the greatest integral value not exceeding \((10 \times 20)/5 = 40\). (In this case we happen to get an integral value). Hence, the maximum-likelihood estimate of \( N \) is 40.

2. Problem 2.45 We need to solve the equation \( x_1 + x_2 + x_3 = 7 \), with the restrictions that \( 1 \geq x_i \geq 6 \). If we rewrite \( y_i = x_i + 1 \), now with \( 0 \geq y_i \geq 5 \) we have: \( y_1 + y_2 + y_3 = 4 \). The upper bound is satisfied automatically. If we view the above equation as four 1s that must be distributed in 4 distinct slots, we can see that the problem is equivalent
with the problem of choosing 4 out of 3 distinct objects with repetition allowed. Thus, the number of ways that 3 tosses of die can sum to 7 is \( \binom{3-1+4}{4} = 15 \). Hence, the required probability is \( P[\text{sum} = 7] = \frac{15}{6^3} = 0.0694 \).

3. Problem 2.93

(a) The board will be working if only one of the 8 chips fails (i.e., 7 out of 8 chips work) or if all 8 of them work (in which case the back-up chip won’t be used). Thus, the probability of a working board is: \( P_b = \binom{8}{7}p^7(1-p) = 1 - p^7(8 - 7p) \).

(b) Let \( A \) be the event that at least one board is working. As in other similar problems, we will work with the complementary event, \( A^c = \) no board is working. Let \( P_F = 1 - P_b = 1 - p^7(8 - 7p) \) be the probability of a board failing. Since we assume that the boards fail independently, for \( n \) boards we get:

\[
P[A^c] = P_F^n = [1 - p^7(8 - 7p)]^n
\]

What we want is

\[
P[A] \geq 0.999
\]

\[
1 - P[A^c] \leq 0.999
\]

\[
P[A^c] \geq 0.001
\]

\[
(1 - p^7(8 - 7p))^n \leq 0.001
\]

\[
n \geq \frac{-3}{\log_{10}(1 - p^7(8 - 7p))}
\]

4. A football team consists of 20 black and 20 white players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what’s the probability that there are no white-black roommate pairs?

There are \( \frac{40!}{2!20!} \) ways of dividing the 40 players into 20 ordered pairs of two each. That’s, there are \( \frac{40!}{2!20!} \) ways of dividing the players into a first pair, a second pair and so on. Hence there are \( \frac{40!}{2!20!20!} \) ways of dividing the players into (unordered) pairs of 2 each. Furthermore, since a division will result in no white-black pairs if the blacks (and
whites) are paired among themselves, if follows that there are \( \left( \frac{20!}{2!} \right)^2 \) such divisions. Hence the probability of no-white-black roommate pairs is given by

\[
\left( \frac{20!}{2!} \right)^2 \frac{40!}{2!^{20!}}
\]

5. An urn contains \( N \) white and \( M \) black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that (1) exactly \( n \) draws are needed, and (2) at least \( k \) draws are needed?

If we let \( X \) denote the number of draws needed to select a black ball, then \( P[X = n] = (1 - p)^{n-1}p \) with \( p = M/(M + N) \). Hence

1. \( P[X = n] = \left( \frac{N}{M+N} \right)^{n-1} \frac{M}{M+N} = \frac{MN^{n-1}}{M+N} \)

2. \[
P[X \geq k] = \frac{M}{M+N} \sum_{n=k}^{\infty} \frac{N}{M+N}^{n-1}
= \left( \frac{N}{M+N} \right)^{k-1}
\]

We can also obtain this directly. Since the probability that at least \( k \) trials are necessary to obtain a success is equal to the probability that the first \( k - 1 \) trials are all failures. That’s, for a geometric random variable, \( P[X \geq k] = (1 - p)^{k-1} \).

6. A certain organism possesses a pair of each of 5 different genes (which we will designate by the first 5 letters of the English alphabet). Each gene appears in 2 forms (which we designate by lowercase and capital letters). The capital letter will be assumed to be the dominant gene in the sense that if an organism possesses the gene pair \( xX \), then it will outwardly have the appearance of the \( X \) gene. For instance, if \( X \) stands for brown eyes and \( x \) for blue eyes, then an individual having either gene pair \( XX \)
Table 1: All possible gene combinations.

<table>
<thead>
<tr>
<th>aA</th>
<th>bB</th>
<th>Cc</th>
<th>Dd</th>
<th>Ee</th>
</tr>
</thead>
<tbody>
<tr>
<td>aa</td>
<td>bb</td>
<td>cc</td>
<td>dD</td>
<td>ee</td>
</tr>
<tr>
<td>aa</td>
<td>bB</td>
<td>cc</td>
<td>dd</td>
<td>ee</td>
</tr>
<tr>
<td>Aa</td>
<td>Bb</td>
<td>Cc</td>
<td>DD</td>
<td>Ee</td>
</tr>
<tr>
<td>Aa</td>
<td>BB</td>
<td>Cc</td>
<td>Dd</td>
<td>Ee</td>
</tr>
</tbody>
</table>

or xX will have brown eyes, whereas one have gene pair xx will be blue eyed. The characteristic appearance of an organism is called its phenotype, whereas its genetic constitution is called its genotype. (Thus 2 organisms with respective genotypes aA, bB, cc, dD, ee and AA, BB, cc, DD, ee would have different genotypes but the same phenotype.) In a mating between 2 organisms each one contributes, at random, one of its gene pairs of each type. The 5 contributions of an organism (one of each of the 5 types) are assumed to be independent and are also independent of the contributions of its mate. In a mating between organisms having genotypes aA, bB, cC, dD, eD and aa, bB, cc, Dd, ee what’s the probability that the progeny will (1) phenotypically, (2) genotypically resemble
(a) the first parent;
(b) the second parent;
(c) either parent;
(d) neither parent?

(1) phenotypically resemble
the first parent: \( \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{(2^5)^2} = \frac{9}{128} \).
the second parent: \( \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{(2^5)^2} = \frac{9}{128} \).
either parent: Since the two parents have different phenotypes (considering all 5 characteristics together), so we know \( \frac{9}{128} + \frac{9}{128} = \frac{9}{64} \).
neither parent: \( 1 - \frac{9}{64} = \frac{55}{64} \).

(2) genotypeically resemble
the first parent: \( \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{(2^5)^2} = \frac{1}{32} \).
the second parent: \( \frac{2 \cdot 2 \cdot 2 \cdot 2}{(2^2)^2} = \frac{1}{32} \).

either parent: Similarly the two parents have different genotypes too (considering all 5 characteristics together). Then we have \( \frac{1}{32} + \frac{1}{32} = \frac{1}{16} \).

neither parent: \( 1 - \frac{1}{16} = \frac{15}{16} \).

7. We toss a biased coin until 3 heads come up \( P[\text{result} = H] = p \). Find the probability that \( k \) tosses are needed. Prove that \( \sum_{k=3}^{\infty} P[k] = 1 \).

If \( k \) tosses are needed, which means in the first \( k - 1 \) tosses, we get 2 heads, and at the \( k \)th toss we get a head, so it should have the probability \( \binom{k-1}{2}p^2(1-p)^{k-3}p = \binom{k-1}{2}p^3(1-p)^{k-3} \).

8. It can be easily shown that if Mr. X arrives at time \( t \) then if:

(a) \( P[\text{he boardes the train}|t \leq 10] = \frac{t + t_0}{60} \)

(b) \( P[\text{he boardes the train}|60 - t_0 \geq t \geq 10] = \frac{10 + t_0}{60} \)

(c) \( P[\text{he boardes the train}|t \leq 60 - t_0] = \frac{70 - t}{60} \)

Hence,

\[ P[\text{he boardes}] = \int_{t=0}^{60} \frac{1}{60} P[\text{he boardes}|he arrives at t]dt \]

Calculating the integral is easy and is left for the reader.

9. Problem

(a)

\[ P[A] = P[(A \cap B) \cup (A \cap B^c)] = P[A \cap B] + P[A \cap B^c] - P[(A \cap B) \cap (A \cap B^c)] = P[A \cap B] + P[A \cap B^c] = P[A|B]P[B] + P[A|B^c]P[B^c] \]
(b) Let $A$ be the event that the first player wins and $B$ the event that the first player wins at his first turn which is basically the beginning of the game. The probability that the sum of two randomly tossed dice is 7 is equal to $\frac{6}{36}$. Then,


$$= 1 \times \frac{6}{36} + (1 - P[A]) \times (1 - \frac{6}{36})$$

Hence, $P[A] = \frac{1}{2 - \frac{6}{36}}$.

(c) By using the above mentioned argument and substituting $p_1$ in stead of $\frac{6}{36}$ and solving the equation you will get that $P[A] = p = \frac{1}{2 - p_1}$.

(d) Let $A$ be the event that the first player wins. Hence,

$$P[A] = \sum_{k=0}^{\infty} p_1[(1 - p_1)^3]^k$$

$$= \frac{p_1}{1 - (1 - p_1)^3}$$

10. Problem 2.94

(a)

$$P[k \ heads|1] = \binom{3}{k} p_1^k (1 - p_1)^{3-k}$$

$$P[k \ heads|2] = \binom{3}{k} p_2^k (1 - p_2)^{3-k}$$

$$P[k \ heads] = P[k \ heads|1]P[1] + P[k \ heads|2]P[2]$$

$$= \binom{3}{k} [p_1^k (1 - p_1)^{3-k} + p_2^k (1 - p_2)^{3-k}] \times 0.5$$

$$P[1|k \ heads] = \frac{P[k \ heads|1]P[1]}{P[k \ heads]}$$

$$= \frac{\binom{3}{k} p_1^k (1 - p_1)^{3-k} \times 0.5}{\binom{3}{k} [p_1^k (1 - p_1)^{3-k} + p_2^k (1 - p_2)^{3-k}] \times 0.5}$$

(b) If $P[1|k \ heads] \geq P[2|k \ heads]$ then coin 1 is more probable.
(c)

\[
P[k \text{ heads out of } m \text{ tosses}|1] = \binom{m}{k} p_1^k(1 - p_1)^{m-k}
\]

\[
P[k \text{ heads out of } m \text{ tosses}|2] = \binom{m}{k} p_2^k(1 - p_2)^{m-k}
\]

\[
P[k \text{ heads}] = P[k \text{ heads}|1]P[1] + P[k \text{ heads}|2]P[2]
\]

\[
= \binom{m}{k} [p_1^k(1 - p_1)^{m-k} + p_2^k(1 - p_2)^{m-k}] \times 0.5
\]

\[
P[1|k \text{ heads}] = \frac{P[k \text{ heads out of } m \text{ tosses}|1]P[1]}{P[k \text{ heads}]}
\]

\[
= \frac{\binom{m}{k} p_1^k(1 - p_1)^{m-k} \times 0.5}{\binom{3}{k} [p_1^k(1 - p_1)^{m-k} + p_2^k(1 - p_2)^{m-k}] \times 0.5}
\]

\[
= \frac{p_1^k(1 - p_1)^{m-k}}{p_1^k(1 - p_1)^{m-k} + p_2^k(1 - p_2)^{m-k}}
\]

11. Problem 2.95 has been solved in the class.