1. The amount of time cars are parked in a parking lot follows an exponential probability law with average time of 1 hour. The charge for the parking is $1 for each half-hour or less. Find the probability that a car pays $k$ dollars.

In order to pay $k$ dollars for parking, the car’s parking time in hours (denoted as $PT$) should be

$$\frac{k - 1}{2} \text{ hours} < PT \leq \frac{k}{2} \text{ hours}.$$ 

The parking time follows an exponential probability law with average time of 1 hour, which means that

$$P[PT \geq t] = e^{-t}$$

with the units of $t$ being hours. Using the result of example 2.10 of the textbook (about the probability of a lifetime that follows an exponential law to be in a specific interval) we have:

$$P[\frac{k - 1}{2} \text{ hours} < PT \leq \frac{k}{2} \text{ hours}] = P[PT > \frac{k - 1}{2}] - P[PT > \frac{k}{2}] \Rightarrow$$

$$P[\frac{k - 1}{2} \text{ hours} < PT \leq \frac{k}{2} \text{ hours}] = e^{-\frac{k - 1}{2}} - e^{-\frac{k}{2}} = e^{-\frac{k}{2}} (e^{0.5} - 1) \Rightarrow$$

$$P[\frac{k - 1}{2} \text{ hours} < PT \leq \frac{k}{2} \text{ hours}] \approx 0.6487 \times e^{-\frac{k}{2}}.$$
2. Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Let \(RR, BB,\) and \(RB\) denote, respectively, the events that the chosen card is the all red, all black, or the red-black card. Letting \(R\) be the event that the upturned side of the chosen card is red, we have that the desired probability is obtained by

\[
P(RB|R) = \frac{P(RB \cap R)}{P[R]} = \frac{P[R|RB]P[RB]}{P[R|RR]P[RR] + P[R|RB]P[RB] + P[R|BB]P[BB]}
\]

\[
= \frac{\frac{1}{2} \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3}}
\]

\[
= \frac{1}{3}
\]

Note that, given that we have a red upper side, the probability that we have red or black on the other side is 2/3 and 1/3 respectively and not 1/2 for each event, as some students might expect.

3. Problem 2.54 (The Birthday Problem).

We will solve the general case for a class of \(k\) students. We are asked to find the probability of the event \(A = \text{“two or more students in a class of } k \text{ students have the same birthday”}\). Counting directly the favorable outcomes for \(A\) is tough. A much easier problem is to count the outcomes of the complementary event \(A^c = \text{“no student has the same birthday”}\). As each person can celebrate his or her birthday on any one of 365 days, there are a total of \((365)^k\) possible outcomes. (We are ignoring the possibility
of someone’s having been born on Feb. 29.) Assuming that each outcome is equally likely, we see that the probability for \(A^c\) is

\[
P[A^c] = \frac{(365 \times 364 \times 363 \times \cdots \times (365 - k + 1))}{(365)^k} = \frac{365!}{(365 - k)! \times (365)^k}
\]

The numbers involved in the above formula are really big even for moderate \(k\) to be computed using a calculator. Thus, we will use Stirling’s approximation formula. We have:

\[
P[A^c] = \frac{365^{365} \times e^{(-365)} \times \sqrt{(2\pi \times 365)}}{(365 - k)^{365-k} \times e^{-(365-k)} \times \sqrt{2\pi \times (365 - k)}} \times 365^k
\]

\[
P[A^c] = \frac{365^{365-k} \times e^{-365} \sqrt{2\pi} \times 365^{0.5}}{(365 - k)^{365-k} \times e^{-(365-k)} \times \sqrt{2\pi} \times (365 - k)^{0.5}}
\]

\[
P[A^c] = \frac{365^{365.5-k} \times e^{-365}}{(365 - k)^{365.5-k} \times e^{-(365-k)}}
\]

\[
P[A^c] = \left(\frac{365}{365 - k}\right)^{365.5-k} \times e^{-k}
\]

For \(k = 20\) we have:

\[
P[A^c] = \left(\frac{365}{345}\right)^{345.5} \times e^{-20} \approx 0.588857 \Rightarrow
\]

\[
P[A] = 1 - P[A^c] \Rightarrow P[A] \approx 0.411
\]

We see that the probability that among 20 students at least 2 have the same birthday is a little less than 50%, which is rather surprising. Actually, if there’re \(k = 23\) students, the probability that at least two of them have the same birthday exceeds 50%.

4. Show that if \(A, B, C\) are independent events, then the events \(A^c, B^c\) and \(C^c\) are also independent.
When $A, B, C$ are said to be independent, that means

$$P[A \cap B \cap C] = P[A]P[B]P[C]$$

$$P[A \cap B] = P[A]P[B]$$

$$P[A \cap C] = P[A]P[C]$$

$$P[B \cap C] = P[B]P[C]$$

To prove that $A^c$ and $B^c$ are independent we need to show that $P[A^c \cap B^c] = P[A^c]P[B^c]$.

$$P[A^c \cap B^c] = P[(A \cup B)^c]$$

$$= 1 - P[A \cup B]$$

$$= 1 - P[A] - P[B] + P[A \cap B]$$

$$= (1 - P[A])(1 - P[B])$$

$$= P[A^c]P[B^c]$$

By the same argument we can show that $P[C^c \cap B^c] = P[C^c]P[B^c]$ and $P[A^c \cap C^c] = P[A^c]P[C^c]$. Now we need to show that $P[A^c \cap B^c \cap C^c] = P[A^c]P[B^c]P[C^c]$ to complete the proof.

$$P[A^c \cap B^c \cap C^c] = 1 - P[A \cup B \cup C]$$

$$= 1 - P[A] - P[B] - P[C] +$$

$$+ P[A \cap B] + P[B \cap C] + P[A \cap C] - P[A \cap B \cap C]$$

$$= (1 - P[A])(1 - P[B])(1 - P[C])$$

$$= P[A^c]P[B^c]P[C^c]$$

5. (a) If 4 Americans, 3 Frenchmen and 3 Germans are to be seated randomly in a row, what is the probability that people from the same nationality sit next to each other?
There are $4!$ ways to arrange the Americans among themselves, $3!$ ways to arrange the Frenchmen among themselves, and $3!$ ways to arrange the Germans among themselves. These three groups can also be arranged in $3!$ ways. The probability that people of the same nationality sit next to each other is simply the number of ways these people can be arranged such that they are sitting next to each other divided by the total number of ways that these 10 people can be arranged. Hence the probability is

$$\frac{4! \times 3! \times 3! \times 3!}{10!} = \frac{24 \times 6 \times 6 \times 6}{3628800} = \frac{5184}{3628800} = 0.00143.$$  

(b) How many different letter arrangements can be formed using the letters $P E P P E R$?

We first note that there are $6!$ permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3 $P$’s and the 2 $E$’s are distinguished from each other. However, consider any one of these permutations - for instance, $P_1P_2E_1P_3E_2R$. If we now permute the $P$’s among themselves and the $E$’s among themselves, then the resultant arrangement would still be of the form $P P E P E R$. That is, all $3! \times 2!$ permutations

$$P_1P_2E_1P_3E_2R \quad P_1P_2E_2P_3E_1R$$
$$P_1P_3E_1P_2E_2R \quad P_1P_3E_2P_2E_1R$$
$$P_2P_1E_1P_3E_2R \quad P_2P_1E_2P_3E_1R$$
$$P_2P_3E_1P_1E_2R \quad P_2P_3E_2P_1E_1R$$
$$P_3P_1E_1P_2E_2R \quad P_3P_1E_2P_2E_1R$$
$$P_3P_2E_1P_1E_2R \quad P_3P_2E_2P_1E_1R$$

are of the form $P P E P E R$. Hence there are $\frac{6!}{3!2!} = 60$ possible letter arrangements of the letters $P E P P E R$.

6. An urn contains the numbers 1, 3, 5, 6, 7, 8, 24, 25, 30, and 50. Seven numbers are chosen randomly without replacement. What is the probability the third largest number is 8?
Let $A$ be the required event. We have 10 numbers, so the total number of equiprobable outcomes in our sample space is the number of ways to pick 7 numbers out of 10 numbers, i.e.

$$\binom{10}{7} = \frac{10!}{7!3!} = \frac{10 \times 9 \times 8}{3 \times 2} = 10 \times 3 \times 4 = 120.$$ 

In order for the third largest number to be 8, we need to:

(a) choose number “8” (there are $\binom{1}{1} = 1$ ways to pick 1 number out of 1 numbers)

(b) choose 2 numbers bigger than 8, i.e. choose 2 numbers from the subset (24, 25, 30, 50) (there are $\binom{4}{2} = 6$ ways to pick 2 numbers out of 4 numbers)

(c) choose 4 numbers smaller than 8, i.e. choose 4 numbers from the subset (1, 3, 5, 6, 7) (there are $\binom{5}{4} = 5$ ways to pick 4 numbers out of 5 numbers)

Hence, the required probability is

$$P[A] = \frac{\binom{1}{1}\binom{4}{2}\binom{5}{4}}{\binom{10}{7}} = \frac{1 \times 6 \times 5}{120} = \frac{30}{120} = \frac{1}{4}.$$ 

7. From a group of 5 men and 7 women, how many different committees consisting of 2 men and 3 women can be formed? What if 2 of the women don’t get along and should not be placed in the same committee?

As there are $\binom{5}{2}$ possible groups of 2 men, and $\binom{7}{3}$ possible groups of 3 women, it follows from the basic counting principle that there are $\binom{5}{2}\binom{7}{3} = 350$ possible committees consisting of 2 men and 3 women.

On the other hand, if 2 of the women refuse to serve on the committee together, then as there are $\binom{2}{0}\binom{5}{3}$ possible groups of 3 women not containing either of the 2 feuding women and $\binom{2}{1}\binom{5}{2}$ groups of 3 women containing either of the 2 feuding women, it follows that there are $\binom{2}{0}\binom{5}{3} + \binom{2}{1}\binom{5}{2} = 30$ groups of 3 women not containing both of
the feuding women. Since there are \( C_2^5 \) ways to choose the 2 men, it follows that, in this case, there are \( 30 \times C_2^5 = 300 \) possible committees.

For the second question we can alternatively think about the number of ways that both of the 2 feuding women could be chosen and subtract that from the total number of possible committees. There are \( C_2^2 \) ways to choose both women and then \( C_1^5 \) ways for the third woman to be chosen (from the remaining 5 women). Since the number of ways to choose the men is the same as above, in total there are

\[
C_2^5 C_3^7 - C_2^5 C_2^2 C_1^5 = 350 - 50 = 300
\]

possible committees.