Periodic sequences play an important role in the analysis of discrete-time signals and systems. A special role is also played by the particular complex exponential sequence \( x(n) = e^{j\omega_0 n} \). In this chapter, we introduce the reader to the notion of periodic sequences and highlight some properties of complex exponential sequences.

**Periodic Sequences.**

A sequence is said to be periodic of period \( N \) if \( N \) is the smallest positive integer such that

\[
x(n) = x(n + N) \quad \text{for all} \quad n
\]

In other words, if \( N \) is the smallest positive integer for which the sequence repeats itself. For example, consider the sinusoidal sequence

\[
x(n) = \sin(\omega_0 n + \theta_0)
\]

for some given \( \{\omega_0, \theta_0\} \). In order to verify whether this sequence is periodic or not, we need to find the smallest positive integer that satisfies

\[
\sin(\omega_0 n + \theta_0) = \sin(\omega_0 (n + N) + \theta_0) \quad \text{for all} \quad n,
\]

i.e., if and only if,

\[
\sin(\omega_0 n + \theta_0) = \sin(\omega_0 n + \theta_0 + \omega_0 N) \quad \text{for all} \quad n,
\]

We know from the properties of the sine function that this equality holds if, and only if, there exists an integer \( N \) such that

\[
\omega_0 N = 2k\pi \quad \text{for some integer} \quad k.
\]

That is, if and only if, \( \omega_0 N \) is a multiple of \( 2\pi \). Actually, we need to find the smallest \( N \) that satisfies this equality. The difficulty is that there need not exist an integer value of \( k \) that results in an integer \( N \); in other words, not every sinusoidal sequence is periodic! This is just one of the subtle differences that exist between discrete-time and continuous-time signals. In continuous-time, all sinusoidal signals are periodic. But not in discrete-time.
To continue with the above example, assume $\omega_0 = 5\pi/3$. Then $N$ and $k$ must be related via

$$N = \frac{6}{5} k.$$  

In this case, the smallest integer $k$ that results in an integer $N$ is $k = 5$. It then follows that $N = 6$ and the sequence $\sin\left(\frac{2\pi}{3} n + \theta_0\right)$ repeats itself every 6 samples. We thus say that it is periodic with period $N = 6$.

But what about a sinusoidal sequence with $\omega_0 = \sqrt{2}$? In this case, $N$ and $k$ should be related via:

$$N = \sqrt{2} \pi k.$$  

It is clear that there does not exist any integer $k$ that results in an integer $N$. For this reason, the sequence $\sin\left(\sqrt{2} \cdot t + \theta_0\right)$ is not periodic! [In contrast, the continuous-time signals $\sin(\sqrt{2} \cdot t + \theta_0)$ and $\sin\left(\frac{2\pi}{3} t + \theta_0\right)$ are both periodic, since in continuous-time the period of a signal is allowed to be any positive real number.]

### Complex Exponential Sequences

Consider now the complex exponential sequence

$$x(n) = e^{j\omega_0 n}$$

This sequence will be periodic if we can find a smallest positive integer $N$ such that

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$$

or, equivalently, if we can find a smallest positive integer $N$ such that

$$e^{j\omega_0 N} = 1$$

We know from the properties of the exponential function that this equality holds if, and only if,

$$\omega_0 N = 2k\pi$$

for some integer $k$.

That is, if and only if, $\omega_0 N$ is a multiple of $2\pi$. As in the case of sinusoidal sequences, complex exponential sequences may or may not be periodic.

So assume $\omega_0 = \pi/6$ and, hence, $x(n) = e^{j\pi/6}$. Then, starting say from $n = 0$, the terms of the sequence will be points on the unit circle at the following successive angles (in degrees and relative to the positive horizontal axis):

$$0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300, 330, 0.$$  

This shows that it takes 12 samples before the sequence repeats itself. We therefore have a periodic sequence of period $N = 12$; a polar plot of which is shown in the Fig. 3.1.

### Angular Frequency

In the above example of each period of $x(n) = e^{j\pi/6}$, the sequence covers a phase change of $2\pi$ radians every period. In other words, when the sequence starts repeating itself, its samples would have covered the circle once. Since the period is $N = 12$ samples, we say that the sequence covers $2\pi/12 = \pi/6$ radians per sample. Therefore, the value $\omega_0 = \pi/6$ is a measure of how many radians are covered per sample by the periodic sequence; it is called the **angular frequency** and is measured in radians/sample.

Consider now the sequence $x(n) = e^{-j\pi/6}$ with a negative value for $\omega_0$. One might wonder about the meaning of a negative $\omega_0$? First note that the sequence $e^{-j\pi/6}$ is still
Figure 3.1. Polar plot of the sequence \( x(n) = e^{j \pi n} \) over one period.

Periodic Sequences

Chapter 3

periodic with period \( N = 12 \). Now, however, the terms of the sequence will be points on the unit circle at the following successive angles (in degrees and relative to the positive horizontal axis):

\[
0, -30, -60, -90, -120, -150, -180, -210, -240, -270, -300, -330, 0.
\]

These terms cover the unit circle in a clockwise direction. This is in contrast to the earlier sequence \( e^{j \frac{\pi}{6} n} \), with positive \( \omega_o \), whose terms cover the circle in a counter-clockwise direction. Therefore, the sign of \( \omega_o \) indicates in which direction the unit circle is covered by the samples of the sequence. We say that the sequence \( x(n) = e^{-j \frac{\pi}{6} n} \) has a negative angular frequency of \( -\pi/6 \) radians/sample, while the sequence \( x(n) = e^{j \frac{\pi}{6} n} \) has a positive angular frequency of \( \pi/6 \) radians/sample. Usually, both sequences are said to have an angular frequency of \( \pi/6 \) radians/sample without being specific about whether the sequence is covering the unit circle in one direction or in the other.

**Euler’s Relation**

How do negative values for \( \omega_o \) arise? One answer lies in the so-called Euler’s relation, which states that

\[
e^{j\omega_o n} = \cos(\omega_o n) + j\sin(\omega_o n)
\]

That is, Euler’s relation expresses a complex exponential in terms of cosine and sine sequences. It follows from Euler’s relation that any real-valued cosine sequence can be expressed as a combination of two exponential sequences, namely,

\[
\cos(\omega_o n) = \frac{1}{2} \left[ e^{j\omega_o n} + e^{-j\omega_o n} \right]
\]
Likewise, every real-valued sine sequence can be expressed as a combination of two exponential sequences, namely,

\[ \sin(\omega_n) = \frac{1}{2j} \left( e^{j\omega_n} - e^{-j\omega_n} \right) \]

We therefore find that a cosine sequence can be obtained by properly combining the terms of two exponential sequences: one with a negative angular frequency and the other with a positive angular frequency. Later in the book, we shall arrive at a similar conclusion for generic sequences \( x(n) \), namely, that under some mild assumptions, a generic sequence \( x(n) \) can be expressed as a linear combination of exponential sequences with positive and negative angular frequencies.

**Relation Between Angular Frequency and Period of a Sequence**

Now that we have introduced the concept of angular frequencies, let us indicate some additional subtleties that arise when studying periodic sequences (in contrast to periodic continuous-time signals).

Let \( \omega_0 = \pi/3 \) and consider the exponential sequence \( x(n) = e^{j\pi/3}n \). This is a periodic sequence with period \( N = 6 \) and \( k = 1 \). The value of \( k \) signifies that the sequence covers a \( 2\pi \) phase change (i.e., a single rotation around the circle) every six samples. The resulting angular frequency is therefore \( 2\pi/6 = \pi/3 \) radians per sample, which is equal to \( \omega_0 \). Note that the angular frequency \( (\pi/3) \) in this example is larger than the angular frequency in the previous example \( (\pi/6) \) and that the period \( (N = 6 \text{ in this example}) \) is correspondingly smaller \( (N = 12 \text{ before}) \).

A word of caution is now in place. Does it always hold that higher angular frequencies imply lower periods? The answer, for discrete-time signals, is negative!

Let \( \omega_0 = 3\pi/4 \) and consider the exponential sequence \( x(n) = e^{j3\pi/4}n \). The value of \( \omega_0 \) is higher than in the previous two cases, \( \{\pi/6, \pi/3\} \). The sequence is still periodic with \( N = 8 \) and \( k = 3 \). The value of \( k \) means that the sequence goes around the circle 3 times before repeating itself. It therefore covers \( 3 \times 2\pi = 6\pi \) radians every period or every 8 samples. This corresponds to an angular frequency of \( 6\pi/8 = 3\pi/4 \) radians/sample. The sequence has therefore higher angular frequency than in the two earlier examples (with \( \omega_0 = \pi/6 \) and \( \omega_0 = \pi/3 \)). Its period, however, is smaller than the period of one of them and larger than the period of the other!

We therefore conclude that we must always compare the (absolute) angular frequencies (radians/sample), and not the periods, in order to determine which sequence has higher angular speed (i.e., which sequence covers more radians per sample).

Moreover, for values of \( \omega_0 \) in the range \([-\pi, \pi]\), the higher the absolute value of \( \omega_0 \) the higher the (absolute) angular frequency of the sequence. In fact, \( \omega_0 \) itself is the angular frequency. This was the case in the examples considered so far.

However, for values of \( \omega_0 \) outside the interval \([-\pi, \pi]\), it does not hold that the higher the (absolute value of) \( \omega_0 \) the higher the angular frequency of the sequence! This is yet another distinction from continuous-time, where the higher the value of the angular frequency (in radians per second) the faster the oscillations of the signal. We clarify this point in the following.

**Indistinguishable Sequences**

Note that

\[ e^{j\omega_n} = e^{j(\omega_0 + 2\pi m)n} \] for all integers \( n \) and \( m \).
That is, two complex exponential sequences whose angular frequencies differ by multiples of $2\pi$ are indistinguishable. This is distinct from the continuous-time case, where it does not hold that $e^{j\Omega t} = e^{j(\Omega + 2\pi m)t}$ for all $t$.

Therefore, if a complex exponential sequence has a value for $\omega_o$ in the range $[\pi, 2\pi]$, then by subtracting $2\pi$ from it we obtain a new value $\omega'_o$ that lies between $[-\pi, 0]$,

$$\omega'_o = \omega_o - 2\pi.$$ 

Both sequences $e^{j\omega_o n}$ and $e^{j\omega'_o n}$ will be indistinguishable. In this case, we choose the smaller number (in absolute value), $\omega'_o$, to be the angular frequency for $e^{j\omega'_o n}$.

In a similar fashion, if an exponential sequence has a value for $\omega_o$ in the range $[-2\pi, -\pi]$, then by adding $2\pi$ to it we obtain a new value $\omega'_o$ that lies between $[0, \pi]$,

$$\omega'_o = \omega_o + 2\pi.$$ 

Both sequences $e^{j\omega_o n}$ and $e^{j\omega'_o n}$ will again be indistinguishable. In this case, we also take the smaller number (in absolute value), $\omega'_o$, to be the angular frequency for $e^{j\omega'_o n}$.

In summary, given an exponential sequence $x(n) = e^{j\omega_o n}$, we first reduce the value of $\omega_o$ to lie within the interval $[-\pi, \pi]$, by adding to, or subtracting from, $\omega_o$ whatever multiples of $2\pi$ that are needed. The new equivalent sequence, $x(n) = e^{j\omega'_o n}$, becomes our starting point for any subsequent analysis. So consider the sequence $x(n) = e^{j\frac{5\pi}{4}}n$. It is periodic with $N = 8$ and $k = 5$. That is, it repeats itself every 8 samples and during one period it covers the circle 5 times (in an anti-clock wise fashion). Hence, it covers $10\pi$ radians per 8 samples, which amounts to $5\pi/4$ radians per sample. However, note that

$$\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

and, hence,

$$e^{j\frac{5\pi}{4}} = e^{-j\frac{3\pi}{4}}$$

The new sequence $y(n) = e^{-j\frac{3\pi}{4}}n$ is again periodic with period $N = 8$ but with $k = 3$. That is, it repeats itself every 8 samples and during one period it covers the circle 3 times (in an clock-wise fashion). Hence, it covers $-6\pi$ radians per 8 samples, which amounts to $-3\pi/4$ radians per sample.

The samples of $y(n)$ coincide with those of $x(n)$. That is, $x(n)$ and $y(n)$ are identical sequences. The question then is which angular frequency should we adopt for $x(n)$? As mentioned above, our convention throughout this book will be to adjust the angular frequencies so that they always lie within the interval $[-\pi, \pi]$. Therefore, for the example at hand, we shall say that the angular frequency of the sequence $x(n) = e^{j\frac{5\pi}{4}}n$ is $-3\pi/4$ radians/sample.

From the above discussion, we conclude that in discrete-time signal processing, the range of values for the angular frequency $\omega_o$ are always limited to a $2\pi$ interval, say $-\pi \leq \omega_o \leq \pi$.

This means that we need not consider complex exponential periodic sequences with angular frequencies outside this range. This is because we can always reduce an angular frequency by integer multiples of $2\pi$ and get an identical sequence with angular frequency within the $[-\pi, \pi]$ range.
Aliases

Angular frequencies $\omega_0$ and $\omega_1$ that differ by multiples of $2\pi$ are called aliases of each other,

$$\omega_0 = 2\pi k \pm \omega_1, \quad -\infty < k < \infty,$$

since they generate identical complex exponential sequences (with same periodicity and apart from a possible sign change).

High and Low Frequencies

We shall say that angular frequencies close to $\pm \pi$ are high frequencies, while angular frequencies close to 0 are low frequencies.
PROBLEMS

Problem 3.1: Plot the sequence \( x(n) = \delta(n + 1) + (\frac{1}{2})^n u(n - 3) \).

Problem 3.2: Answer either True or False to the following statements and give brief justifications.

<table>
<thead>
<tr>
<th>Statement</th>
<th>T</th>
<th>F</th>
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<tbody>
<tr>
<td>01. The sequence ( u(n) - u(n - 3) ) has only 3 nonzero samples.</td>
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</tr>
<tr>
<td>02. The sequence ( e^{j\frac{\pi}{3}n} ) has period ( N = 5 ) samples.</td>
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<tr>
<td>03. The samples of the sequence ( e^{j\omega_0 n} ) have unit magnitude for all ( n ).</td>
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<td></td>
</tr>
<tr>
<td>04. The frequency of ( \sin \pi t ) is 0.5 Hz or ( \pi \text{ rad/sec} ).</td>
<td></td>
<td></td>
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<tr>
<td>05. This sinusoid covers a 2( \pi ) phase change per period.</td>
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<tr>
<td>06. Sampling it every 0.3 sec leads to the</td>
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<tr>
<td>sequence ( \sin \frac{0.3\pi n}{3} ) of period ( N = 20 ) samples.</td>
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<tr>
<td>07. This sequence also covers 2( \pi ) phase change per period.</td>
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<td></td>
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<tr>
<td>08. ( \cos 0.1\pi n ) and ( \cos 1.9\pi n ) have the same angular frequency.</td>
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</tbody>
</table>

Problem 3.3: Let \( x(n) \) be a periodic sequence of period \( N \) and assume its energy over a period is equal to \( K \). Show that its average power is equal to \( K/N \).

Problem 3.4: What is the period of the sequence \( x(n) = \sin \left( \frac{n}{3} + \frac{\pi}{4} \right) \)?

1. 3. 6\( \pi \).
2. 6.
3. None of the above.

Problem 3.5: Is the sequence \( e^{j\frac{\omega_0}{3} n} + e^{j\frac{\omega_1}{3} n} \) periodic? If so, what is its period? Determine also its energy and average-power.

Problem 3.6: Assume \( x(n) \) has period \( N \). Are the following sequences periodic? If so, determine their periods in terms of \( N \):

(i) \( x(1 - 2n) \)?
(ii) \( x(n) + (-1)^n x(0) \)?

Problem 3.7: If \( x(n) \) is periodic, prove that \( e^{j\frac{\omega}{N} n} x(n) \) is also periodic no matter what the period of \( x(n) \) is.

Problem 3.8: Prove that \( \sin(\omega n) \) is periodic if, and only if, \( \frac{\omega}{\pi} \) is a rational number.

Problem 3.9: True or False? Except for the zero sequence, every periodic sequence has infinite energy.

Problem 3.10: True or False? The period of the sum of two periodic sequences is always the least-common multiple of their periods.

Problem 3.11: Consider the sequence

\[ x(n) = e^{j\frac{\omega_0}{N} n} + e^{j\frac{\omega_1}{N} n}. \]

Show that it can be written in the form

\[ x(n) = Ae^{j\omega_0 n} \cos(\omega_1 n), \]

for some positive real number \( A \), and for some \( \omega_0 > \omega_1 \). Is \( x(n) \) periodic?