Mutually Coupled Inductors

\[ N_1 \text{ and } N_2 \text{ are number of turns in coils 1 and 2.} \]
\[ \text{Current } i_1 \text{ in coil 1 produces magnetic flux } \phi_{11} \text{ in coil 1:} \]
\[ \phi_{11}(t) = P_{11} N_1 i_1(t), \]
where \( P_{11} \) is proportionality constant.

Some of flux \( \phi_{11} \) will generate flux \( \phi_{21} \) in nearby coil 2:
\[ \phi_{21}(t) = P_{21} N_1 i_1(t). \]

Similarly, current \( i_2 \) in coil 2 will generate flux in both coils:
\[ \phi_{12}(t) = P_{12} N_2 i_2(t), \]
\[ \phi_{22}(t) = P_{22} N_2 i_2(t). \]

For linear elements, \( P_{12} \) must equal \( P_{21} \).

Total flux in coil 1 = \( \phi_1 = \phi_{11} + \phi_{12} \).
Total flux in coil 2 = \( \phi_2 = \phi_{21} + \phi_{22} \).

Faraday’s Law:
\[ v(t) = N \frac{d\phi(t)}{dt} \]
so,
\[ v_1(t) = N_1 [P_{11} N_1 di_1/dt + P_{12} N_2 di_2/dt], \]
\[ v_2(t) = N_2 [P_{12} N_1 di_1/dt + P_{22} N_2 di_2/dt]. \]
Define:
\[ L_1 = P_{11} N_1^2 \text{ (self-inductance of coil 1)} \]
\[ L_2 = P_{22} N_2^2 \text{ (self-inductance of coil 2)} \]
And
\[ M = P_{12} N_1 N_2 \text{ (mutual inductance)} \]
Then
\[ v_1(t) = L_1 di_1/dt + M di_2/dt, \]
\[ v_2(t) = M di_1/dt + L_2 di_2/dt. \]

Sign of mutual inductance term depends on whether fluxes add or subtract.

**Dot convention**

Both currents into or out of dots, fluxes add \( \rightarrow \) sign of \( M \) is positive.
One current into dot, other current out of dot, fluxes subtract \( \rightarrow \) sign of \( M \) is negative.
If currents are always defined into dots, voltages always defined with plus at dot, then mutual inductance term has positive coefficient.

**Sinusoidal Signals**

Use phasor convention:
\[ v(t) = V e^{j\omega t}. \]
\[ i(t) = I e^{j\omega t}. \]
then,
\[ V_1 e^{j\omega t} = L_1 \frac{d(I_1 e^{j\omega t})}{dt} + M \frac{d(I_2 e^{j\omega t})}{dt}, \]
\[ V_2 e^{j\omega t} = M \frac{d(I_1 e^{j\omega t})}{dt} + L_2 \frac{d(I_2 e^{j\omega t})}{dt}. \]
So
\[ V_1 = j\omega L_1 I_1 + j\omega M I_2, \]
\[ V_2 = j\omega M I_1 + j\omega L_2 I_2. \]
where \( L_1 \geq 0, L_2 \geq 0, M \geq 0. \)

**Example 4.1 Mutual inductance.**
Find current in coil 2:
Always use mesh method for mutual inductance problems.
In coil 1: \( I_1 = I_A - I_B \)
In coil 2: \( I_2 = I_B \)
Define voltages across coils, positive at dots:

\[ V_1 = j\omega L_1 I_1 + j\omega M I_2 = j1 (I_A - I_B) + j1 I_B. \]
\[ V_2 = j\omega M I_1 + j\omega L_2 I_2 = j1 (I_A - I_B) + j2 I_B. \]

KVL:
Loop 1:
\[ V_S = 2 I_A + j1 (I_A - I_B) + j1 I_B + (I_A - I_B) \]
Loop 2:
\[ 0 = (I_A - I_B) + j1 (I_A - I_B) + j1 (I_A - I_B) - j2 I_B - 3 I_B. \]
Combining:
\[ V_S = (3 + j1) I_A - I_B \]
\[ 0 = I_A - (4 + j1) I_B. \]

2 equations, 2 unknowns. Substitute:
\[ I_A = (4 + j1) I_B. \]

Then
\[ V_S = (3 + j1) (4 + j1) I_B - I_B = (12 + j4 + j3 - 1) I_B = (11 + j7) I_B. \]

So
\[ I_B = V_S / (11 + j7) \]

END EXAMPLE 4.1.

Energy in Coupled Inductors

Total energy stored in inductors \( W(t) = \int_{-\infty}^{t} p(\tau) \, d\tau \)

Assume currents increase from 0 to \( i_1, i_2 \) in inductors 1 and 2.

Instantaneous power \( p(t) = i_1 v_1 + i_2 v_2. \)

Then
\[ W(t) = \int_{-\infty}^{t} p(\tau) \, d\tau = \int_{-\infty}^{t} (i_1 v_1 + i_2 v_2) \, d\tau = \int_{-\infty}^{t} (i_1 L_1 di_1/d\tau + i_1 M di_2/d\tau + i_2 M di_1/d\tau + i_2 L_2 di_2) \, d\tau = \int_{-\infty}^{t} (L_1 i_1 + M d(i_1 i_2) + L_2 i_2) \, d\tau = 0.5 L_1 i_1^2 + M i_1 i_2 + 0.5 L_2 i_2^2. \]

assuming currents are 0 at \(-\infty\).

Find minimum energy in coils for given value of \( i_1. \)
\[ dW/di_2 = 0 = L_2 i_2 + M i_1. \]

So
\[ i_2 = - (M / L_2) i_1. \]

Substitute in \( W: \)
\[ W_{\text{min}} = 0.5 L_1 i_1^2 - M i_1 (M / L_2) i_1 + 0.5 L_2 ((M / L_2) i_1)^2 = 0.5 L_1 i_1^2 - (M^2 / L_2) i_1^2 + 0.5 (M^2 / L_2) i_1^2 = i_1^2 (0.5 L_1 - 0.5 (M^2 / L_2)). \]

Since minimum energy must be \( \geq 0, \)
\[ L_1 - (M^2 / L_2) \geq 0, \]
Or
\[ M \leq \sqrt{L_1 L_2}. \]

Coefficient of coupling

Define
\[ k = M / \sqrt{L_1 L_2} \]

so
0 \leq k \leq 1.

k = 0: No coupling.
k = 1: Perfect coupling.

**Example 4.2. Coefficient of coupling.**
If \( L_1 = 10 \text{ mH}, M = 6 \text{ mH}, k = 1 \), find \( L_2 \).
\[
L_2 = \frac{M^2}{(L_1 k^2)} = \frac{36}{10} = 3.6 \text{ mH}.
\]
END EXAMPLE 4.2.

**Models for Mutual Inductance.**
Model 1: (always correct)
\[
V_1 = j\omega L_1 I_1 + j\omega M I_2,
V_2 = j\omega M I_1 + j\omega L_2 I_2.
\]

Model 2:  
If bottoms of inductors are shorted:
\[
V_1 = j\omega L_A I_1 + j\omega L_C (I_1 + I_2)
= j\omega (L_A + L_C) I_1 + j\omega L_C I_2
V_2 = j\omega L_B I_2 + j\omega L_C (I_1 + I_2)
= j\omega L_C I_1 + j\omega (L_B + L_C) I_2
\]

Compare with
\[
V_1 = j\omega L_1 I_1 + j\omega M I_2,
V_2 = j\omega M I_1 + j\omega L_2 I_2.
\]

Thus,
\[
L_1 = L_A + L_C
L_2 = L_B + L_C
M = L_C
\]
or,
\[
L_A = L_1 - M
L_B = L_2 - M
L_C = M.
\]

Note: \( L_A \) or \( L_B \) can be negative: Not realizable, just mathematical model.

When can we use Model 2?  
Consider:
Connection makes NO difference if loops are not connected (current in connector must be zero if no return path) – OK to connect, use model 2. But if loops are already connected, this creates a new loop, not OK.

Consider circuit:

\[
\begin{array}{c}
\text{L}_1 \quad \text{R} \quad \text{L}_2 \\
\end{array}
\]

Can reposition elements, same circuit:

\[
\begin{array}{c}
\text{R} \quad \text{L}_2 \quad \text{L}_1 \\
\end{array}
\]

Then use Model 2:

\[
\begin{array}{c}
\text{R} \quad \text{L}_A \quad \text{L}_B \\
\text{L}_C \\
\end{array}
\]

Since no current flows in \(L_C\), remove from circuit:

\[
\begin{array}{c}
\text{R} \quad \text{L}_1 - \text{M} \quad \text{L}_2 - \text{M} \\
\end{array}
\]

**Reflected Impedance**

Use mesh method, with Model 1:

\[
\begin{align*}
V_S &= Z_1 I_1 + j\omega L_1 I_1 + j\omega M I_2, \quad (1) \\
0 &= Z_2 I_2 + j\omega L_2 I_2 + j\omega M I_1. \quad (2)
\end{align*}
\]

From (2):

\[
I_2 = -\frac{I_1 j\omega M}{Z_2 + j\omega L_2}.
\]

Substitute in (1):

\[
V_S = (Z_1 + j\omega L_1) I_1 + j\omega M \left[- I_1 j\omega M / (Z_2 + j\omega L_2)\right],
\]

So

\[
Z_{in} = \frac{V_S}{I_1} = Z_1 + j\omega L_1 - (j\omega M)^2 / (Z_2 + j\omega L_2)
\]

\[
= Z_1 + j\omega L_1 + \omega^2 M^2 / (Z_2 + j\omega L_2).
\]

Primary impedance \(Z_{pri} = Z_1 + j\omega L_1\)

Secondary impedance \(Z_{sec} = \omega^2 M^2 / (Z_2 + j\omega L_2)\).

Consider secondary impedance:

Let \(Z_2 = R_2 + j X_2\), then

\[
Z_{sec} = \frac{\omega^2 M^2}{(Z_2 + j\omega L_2)}
\]

\[
= \frac{\omega^2 M^2}{(R_2 + j (X_2 + \omega L_2))}
\]

\[
= \frac{(\omega^2 M^2 R_2 - j\omega^2 M^2 (X_2 + \omega L_2)) / (R_2^2 + (X_2 + \omega L_2)^2)}.
\]

Note: Re\([Z_{sec}] > 0 \rightarrow \text{real resistance.}

\[
\text{Im}[Z_{sec}] = -j\omega^2 M^2 (X_2 + \omega L_2) / (R_2^2 + (X_2 + \omega L_2)^2).
\]

If \(X_2 \geq 0\) (i.e., inductor), then \(\text{Im}[Z_{sec}] \leq 0\) (capacitor).
If $X_2 < -\omega L_2$ (i.e., capacitor), then $\text{Im}[Z_{\text{sec}}] > 0$ (inductor).

If $-\omega L_2 < X_2 < 0$, (i.e., capacitor), then $\text{Im}[Z_{\text{sec}}] \leq 0$ (capacitor).

**Ideal Transformer**

![Ideal Transformer Diagram]

Ideal transformer behaves like 2 mutual inductors with perfect coupling, infinite inductance.

Perfect coupling $\rightarrow \phi_1 = \phi_1 = \phi$.

Then

$v_1(t) = N_1 d\phi_1 / dt = N_1 d\phi / dt$

$v_2(t) = N_2 d\phi_2 / dt = N_2 d\phi / dt$

so

$v_1(t) / v_2(t) = N_1 / N_2$.

Also, no power dissipation (pure inductance), so

$i_1 v_1 + i_2 v_2 = 0$,

thus,

$i_2 / i_1 = -N_1 / N_2$.

**Example 4.3. Reflected impedance.**

![Reflected Impedance Diagram]

Find input impedance

$Z_{\text{in}} = V_S / I_{\text{in}}$.

$V_1 = V_S \Rightarrow V_2 = nV_1 = nV_S$

and

$I_{\text{in}} = I_1 \Rightarrow I_2 = -I_1 / n = -I_{\text{in}} / n$.

Also,

$I_2 = -V_2 / Z_L = -nV_S / Z_L$

Thus,

$I_{\text{in}} = -nI_2 = -n(-nV_S / Z_L) = n^2 V_S / Z_L$.

So

$Z_{\text{in}} = V_S / I_{\text{in}} = Z_L / n^2$.

Can replace transformer and $Z_L$ by $Z_L / n^2$.

END EXAMPLE 4.3.

**Example 4.4. Primary side transformation.**

![Primary Side Transformation Diagram]

Find Thevenin equivalent circuit.

Open circuit: Find $V_{OC}$

$I_2 = 0 \Rightarrow I_1 = 0$.

Then

$V_S = I_1 Z_S + V_1 = V_1$.

And

$V_{OC} = V_2 = nV_1 = nV_S$.

Short circuit: Find $I_{SC}$:

$V_2 = 0 \Rightarrow V_1 = 0$.

Then

$V_S = I_1 Z_S + V_1 = I_1 Z_S \Rightarrow I_1 = V_1 / Z_S$. 

So
\[ I_{SC} = -I_2 = \frac{I_1}{n} = \frac{V_S}{nZ_S}. \]
Finally,
\[ Z_T = \frac{V_{oc}}{I_{SC}} = \frac{nV_S}{(V_S/nZ_S)} = n^2Z_S. \]
So equivalent circuit becomes:

If dots are opposite, sign of \( n \) changes → voltage changes polarity, no change in impedances.

**END EXAMPLE 4.4.**

**Derivation of Ideal Transformer**

Why is there no inductance associated with ideal transformer?

Consider ideal transformer as mutually coupled inductors.

\[ V_S = j\omega L_1 I_1 + j\omega M I_2, \quad (1) \]
\[ 0 = Z_L I_2 + j\omega L_2 I_2 + j\omega M I_1. \quad (2) \]

From (2):
\[ I_2 = -\frac{I_1 j\omega M}{(Z_L + j\omega L_2)}. \]
Substitute in (1):
\[ V_S = I_1 (j\omega L_1 - (j\omega M)^2 / (Z_L + j\omega L_2)) = I_1 (j\omega L_1 + \omega^2 M^2 / (Z_2 + j\omega L_2)). \]

So
\[ Z_{in} = \frac{V_S}{I_1} = j\omega L_1 + \omega^2 M^2 / (Z_L + j\omega L_2) = j\omega L_1 + \omega^2 M^2 (Z_L - j\omega L_2) / (Z_L^2 + \omega^2 L_2^2) = \omega^2 M^2 Z_L / (Z_L^2 + \omega^2 L_2^2) + j\omega [L_1 - L_2 \omega^2 M^2 / (Z_L^2 + \omega^2 L_2^2)]. \]

Assume ideal coupling: \( M = \sqrt{L_1 L_2} \).
\[ Z_{in} = \omega^2 L_1 L_2 Z_L / (Z_L^2 + \omega^2 L_2^2) + j\omega [L_1 - \omega^2 L_1 L_2^2 / (Z_L^2 + \omega^2 L_2^2)]. \]

Let \( L_2 = n^2 L_1 \):
\[ Z_{in} = \omega^2 n^2 L_1^2 Z_L / (Z_L^2 + \omega^2 n^4 L_1^2) + j\omega [L_1 - \omega^2 n^4 L_1^3 / (Z_L^2 + \omega^2 n^4 L_1^2)]. \]

Let \( L_1 \rightarrow \infty \):
\[ Z_{in} \rightarrow \omega^2 n^2 L_1^2 Z_L / (\omega^2 n^4 L_1^2) + j\omega [L_1 - \omega^2 n^4 L_1^3 / (\omega^2 n^4 L_1^2)] = Z_L / n^2 + j\omega [L_1 - L_1] = Z_L / n^2. \]

So input impedance includes no inductance from ideal transformer.

**Example 4.5. Circuit with ideal transformer.**

Let \( V_S = 36 \angle 0 \text{ deg} \ V. \)
\[ Z_{R1} = 3, \quad Z_{R2} = 6, \]
\[ Z_{R3} = 2, \quad Z_{R4} = 2, \]
\[ Z_{C1} = -j2, \quad Z_{C2} = -j2, \]
\( n = 2. \)

Find \( V_O \):
Transform source side (multiply voltage by n, negative because dots reversed, multiply impedances by n²)

\[ V_{OC} = (-n \cdot V_S) \cdot \left( \frac{n^2 R_2}{n^2 R_1 + n^2 R_2} \right) \]
\[ = (-72) \cdot \left( \frac{24}{12 + 24} \right) \]
\[ = -48 \]
\[ = 48 \angle 180 \text{ deg. V.} \]

Find \( Z_T \):
\[ Z_T = Z_{C2} + n^2 Z_{C1} + n^2 R_3 + \left( n^2 R_1 \right) \cdot \left( \frac{n^2 R_2}{n^2 R_1 + n^2 R_2} \right) \]
\[ = -j2 + 4 (-j2) + 4 (2) + (12) (24) / (12 + 24) \]
\[ = -j2 - j8 + 8 \]
\[ = 16 - j10. \]

\[ V_O = \frac{V_T \cdot R_4}{(R_4 + Z_T)} \]
\[ = \frac{(-48) \cdot (2)}{(2 + 16 - j10)} \]
\[ = \frac{-96}{(18 - j10)} \]
\[ = \frac{(96 \angle 180)}{(20.59 \angle -29.05)}. \]
\[ = 4.66 \angle 209.05. \]
\[ = 4.66 \angle -150.95 \text{ deg V.} \]

End Example 4.5

End Chapter 4.

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