Chapter 2. AC Circuit Analysis Techniques

Node and mesh analysis techniques for phasors similar to EE10.

**KVL:** Around each loop,
\[ \sum v_i(t) = 0 \]
\[ \sum V_i e^{j\omega t} = 0 \]
\[ \sum V_i = 0 \]

**KCL:** At each node,
\[ \sum i_i(t) = 0 \]
\[ \sum I_i e^{j\omega t} = 0 \]
\[ \sum I_i = 0 \]

**Example 2.1: Node method.**

![Node diagram]

Let
\[ Z_R = 2, \]
\[ Z_L = j 1, \]
\[ Z_C = -j 2, \]
\[ I_S = 2 \angle 60 \text{ deg A}, \]
\[ V_T = 12 \angle 30 \text{ deg V}. \]

Find \( V_1 \):

\[ V_1 \text{ and } V_2 \text{ related by } \]
\[ V_1 = V_2 + V_T \]

2 node equations.

Apply KCL to Node 1:
\[ I_T + V_1 / Z_L - I_S = 0. \]
Apply KCL to Node 2:
\[ V_2 / Z_C + V_2 / Z_R - I_T = 0. \]
Add node equations to eliminate \( I_T \):
\[ V_2 / Z_C + V_2 / Z_R + V_1 / Z_L - I_S = 0. \]
Substitute for \( V_2 \):
\[ (V_1 - V_T) / Z_C + (V_1 - V_T) / Z_R + V_1 / Z_L - I_S = 0. \]
\[ V_1 (1/Z_C + 1/Z_R + 1/Z_L) = V_T (1/Z_C + 1/Z_R) + I_S. \]
\[ V_1 (Y_C + Y_R + Y_L) = V_T (Y_C + Y_R + Y_L) + I_S. \]

Substitute numerical values:
\[ Y_R = 0.5, \]
\[ Y_L = -j 1, \]
\[ Y_C = j 0.5, \]
\[ I_S = 2 \angle 60 \text{ deg A}, \]
\[ V_T = 12 \angle 30 \text{ deg V}. \]

So
\[ V_1 = (12 \angle 30) (0.5 + j 0.5) / (0.5 - j 0.5) + (2 \angle 60) / (0.5 - j 0.5) \]
\[ = (12 \angle 30) (0.5 \sqrt{2} \angle 45) / (0.5 \sqrt{2} \angle -45) + (2 \angle 60) / (0.5 \sqrt{2} \angle -45) \]
\[ = (12 \angle 120) + (2 \sqrt{2} \angle 105). \]

To add, convert to rectangular notation:
\[ V_1 = 12 (\cos 120 + j \sin 120) + 2 \sqrt{2} (\cos 105 + j \sin 105) \]
\[ = (-6.00 + j 10.39) + (-0.73 + j 2.73) \]
\[ = -6.73 + j 13.12. \]

Convert back to polar notation:
\[ V_1 = 14.75 \angle 117.16 \text{ deg V} \]

END EXAMPLE 2.1.

**Example 2.2. Node method with dependent source.**

Let \( V_T = 12 \angle 10 \text{ deg V}, \)
\[ I_S = 2 \angle 60 \text{ deg A}, \]
\[ Z_R = 2, \]
\[ Z_L = j 2. \]

I_T + V_1 / Z_L - I_S = 0.
Apply KCL to Node 2:
\[ V_2 / Z_C + V_2 / Z_R - I_T = 0. \]
Add node equations to eliminate I_T:
\[ V_2 / Z_C + V_2 / Z_R + V_1 / Z_L - I_S = 0. \]
Substitute for V_2:
\[ (V_1 - V_T) / Z_C + (V_1 - V_T) / Z_R + V_1 / Z_L - I_S = 0. \]
\[ V_1 (1/Z_C + 1/Z_R + 1/Z_L) = V_T (1/Z_C + 1/Z_R) + I_S. \]
\[ V_1 (Y_C + Y_R + Y_L) = V_T (Y_C + Y_R + Y_L) + I_S. \]

Substitute numerical values:
\[ Y_R = 0.5, \]
\[ Y_L = -j 1, \]
\[ Y_C = j 0.5, \]
\[ I_S = 2 \angle 60 \text{ deg A}, \]
\[ V_T = 12 \angle 30 \text{ deg V}. \]

So
\[ V_1 = (12 \angle 30) (0.5 + j 0.5) / (0.5 - j 0.5) + (2 \angle 60) / (0.5 - j 0.5) \]
\[ = (12 \angle 30) (0.5 \sqrt{2} \angle 45) / (0.5 \sqrt{2} \angle -45) + (2 \angle 60) / (0.5 \sqrt{2} \angle -45) \]
\[ = (12 \angle 120) + (2 \sqrt{2} \angle 105). \]

To add, convert to rectangular notation:
\[ V_1 = 12 (\cos 120 + j \sin 120) + 2 \sqrt{2} (\cos 105 + j \sin 105) \]
\[ = (-6.00 + j 10.39) + (-0.73 + j 2.73) \]
\[ = -6.73 + j 13.12. \]

Convert back to polar notation:
\[ V_1 = 14.75 \angle 117.16 \text{ deg V} \]

END EXAMPLE 2.1.
Dependent source: $2 \ I_0$ where $I_0 = \text{current through } R$.
Find $V_L$.

Apply KCL to supernode:
$$\frac{V_2}{Z_L} + 2 \ I_0 + \frac{V_1}{Z_R} - V_T = 0.$$ 

But
$$I_0 = \frac{V_1}{Z_R}$$
$$V_1 = V_2 + V_T$$

So
$$\frac{(V_1 - V_T)}{Z_L} + 2 \ \frac{V_1}{Z_R} + \frac{V_1}{Z_R} - I_S = 0.$$ 
$$V_1 (1 / Z_L + 3 / Z_R) = \frac{V_T}{Z_L} + I_S.$$ 
Combining,
$$\frac{V_1}{V_T Y_L / (Y_L + 3 Y_R) + I_S / (Y_L + 3 Y_R)} = \frac{(12 \angle 10)}{(1.5 - \ j 0.5) / (2 \angle 60) / (1.5 - \ j 0.5)} = \frac{(12 \angle 10) (0.5 \angle -90)}{(1.58 \angle -18.44) + (2 \angle 60) / (1.58 \angle -18.44)} = \frac{(3.79 \angle -62.56) + (1.26 \angle 78.44)}.$$ 
Convert to rectangular:
$$V_1 = (1.81 - \ j 3.34) + (0.25 + \ j 1.24) = 2.06 - \ j 2.10 = 2.94 \angle -45.51 \ \text{deg V}.$$ 

END EXAMPLE 2.2.

Mesh Method

Example 2.3. Mesh method.

Let $V_S = 24 \angle 0 \ \text{deg V},$
$$I_T = 2 \angle 90 \ \text{deg A},$$
$$Z_{R1} = Z_{R2} = 2,$$
$$Z_L = j 2,$$
$$Z_C = -j 2.$$

Find $I_{R2}$.

Two loops. Choose one loop so that $I_2 = I_T$ (current $I_1$ must not pass through $I_T$). Apply KVL:
$$I_2 = I_T,$$
$$Z_{R1} (I_1 - I_2) + Z_C I_1 + Z_{R2} I_1 - V_S = 0.$$ 
Then
$$Z_{R1} (I_1 - I_T) + Z_C I_1 + Z_{R2} I_1 - V_S = 0,$$
$$I_1 (Z_{R1} + Z_C + Z_{R2}) = I_T Z_{R1} + V_S.$$ 
Then
$$I_1 = I_T Z_{R1} / (Z_{R1} + Z_C + Z_{R2}) + V_S / (Z_{R1} + Z_C + Z_{R2}) = (I_T Z_{R1} + V_S) / (Z_{R1} + Z_C + Z_{R2}) = (2 (2 \angle 90) + 24 \angle 0) / (2 - j 2 + 2) = (24 + j 4) / (4 - j 2) = (24.33 \angle 9.5) / (4.47 \angle -26.5) = 5.44 \angle 36.0 \ \text{deg A}.$$ 
$$I_{R2} = I_1 = 5.44 \angle 36.0 \ \text{deg A}.$$ 

END EXAMPLE 2.3.
Example 2.4. Mesh with dependent current source.

Let $V_S = 24 \angle 0$ deg V,
$Z_{R1} = Z_{R2} = 2,$
$Z_L = j 2,$
$Z_C = -j 2.$

Dependent current source $I_A = 2 \cdot V_A$ where $V_A =$ voltage across $R_1$.
Find $V_0$.

2 loops. Draw one loop so that $I_1 = I_A$. KVL:
$Z_{R1} I_0 + Z_C I_0 + Z_L (I_0 - I_1) - V_S = 0,$
$I_1 = I_A = 2 \cdot V_A$.

Then
$Z_{R1} I_0 + Z_C I_0 + Z_L (I_0 - 2 \cdot V_A) - V_S = 0.$

But
$V_A = Z_{R1} I_0$.

So
$Z_{R1} I_0 + Z_C I_0 + Z_L (I_0 - 2 \cdot Z_{R1} I_0) - V_S = 0,$
$I_0 (Z_{R1} + Z_C + Z_L - 2 \cdot Z_L \cdot Z_{R1}) = V_S,$

So
$I_0 = \frac{V_S}{(Z_{R1} + Z_C + Z_L - 2 \cdot Z_L \cdot Z_{R1})}$
$= \frac{(24 \angle 0)}{(2 - j 2 + j 2 - 2 (j 2) (2))}$
$= \frac{(24 \angle 0)}{(2 - j 8)}$
$= \frac{(24 \angle 0)}{(8.24 \angle -76.0)}$
$= 2.91 \angle 76.0$ deg A.

Then
$V_0 = 2 \cdot V_A \cdot Z_{R2}$
$= 2 \cdot Z_{R1} \cdot Z_{R2} \cdot I_0$
$= 2 \cdot (2) \cdot (2.91 \angle 76.0)$

$= 23.28 \angle 76.0$ deg V.

END EXAMPLE 2.4.

Thevenin’s Theorem.
Any linear circuit can be simplified to one voltage source and one series impedance.
$V_{AB} = V_T + Z_T I_{AB}$

Norton’s Theorem.
Any linear circuit can be simplified to one current source and one parallel impedance.
$I_{AB} = I_N - V_{AB} / Z_N$

Open circuit:
Define $V_{OC} = V_{AB}$ when $I_{AB} = 0$. Then from Thevenin:
\[ V_{OC} = V_T \]

Short circuit:
Define \( I_{SC} = -I_{AB} \) when \( V_{AB} = 0 \). Then from Thevenin:
\[ I_{SC} = \frac{V_T}{Z_T} \]
Substituting,
\[ I_{SC} = \frac{V_{OC}}{Z_T} \]
Or
\[ Z_T = \frac{V_{OC}}{I_{SC}} \]

Similarly, from Norton:
\[ V_{OC} = I_N Z_N \]
And
\[ I_{SC} = I_N \]
Substituting,
\[ V_{OC} = I_{SC} Z_N \]
Or
\[ Z_N = \frac{V_{OC}}{I_{SC}} \]
Note that
\[ Z_T = Z_N \]

**Finding the Thevenin (Norton) impedance: \( Z_T \)**

1. If no dependent sources,
   Set all independent sources to zero (short voltage sources, open current sources), find input impedance \( Z_{in} \), then
   \[ Z_T = Z_{in} \]

2. If both dependent and independent sources,
   Find \( V_{OC} \) and \( I_{SC} \), then
   \[ Z_T = \frac{V_{OC}}{I_{SC}} \]

3. Alternate method:
   Set independent sources = 0, leave dependent source in circuit, apply \( V_{test} \), measure \( I_{test} \), then
   \[ Z_T = \frac{V_{test}}{I_{test}} \]
   This method may be easiest if multiple independent sources.

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**A Simple Example:**

Remove 5-ohm resistor, find \( Z_T \):  
\[ Z_T = 3 + 3 \parallel 6 = 3 + (6) (3) / (3 + 6) = 5. \]

Find \( V_{OC} \): Current only in 3 and 6 ohm resistors:
\[ V_{OC} = 9 (6) / (6 + 3) = 6 \text{ V}. \]

Thevenin equivalent circuit becomes:
Then, put 5-ohm resistor back, use Thevenin equivalent to find $V_0$:

$$V_0 = \frac{V_{OC} \times 5}{5 + Z_T} = 3 \text{ V}.$$

Can also find $I_{SC}$:

$$I_S = \frac{9}{3 + \| 6} = \frac{9}{5} = 1.8 \text{ A}.$$

Then

$$I_{SC} = \frac{I_S \times 6}{(3 + 6)} = 1.2 \text{ A}.$$

Note that $Z_T = \frac{V_{OC}}{I_{SC}}$.

**Example 2.5. Thevenin’s theorem.**

Let $V_1 = 24 \angle 0 \text{ deg V}$,

$$V_2 = 12 \angle 0 \text{ deg V},$$

$$Z_{R1} = Z_{R0} = 2,$$

$$Z_L = j2,$$

Then

$$Z_C = -j2.$$

Find $V_0$ using Thevenin’s theorem.

Analyze circuit without resistor:

Open circuit, find $V_{OC}$:

$$V_1 + V_2 - I \left( Z_{R1} + Z_L \right) = 0.$$

$$I = \frac{V_1 + V_2}{Z_{R1} + Z_L}$$

$$= (24 \angle 0 + 12 \angle 0) / (2 + j2)$$

$$= 9 \sqrt{2} \angle -45 \text{ deg A.}$$

Then

$$V_{OC} = V_1 - I \times Z_{R1}$$

$$= (24 \angle 0) - (9 \sqrt{2} \angle -45) \times 2$$

$$= 24 - 18 \left( 1 - j1 \right)$$

$$= 6 + j18.$$

Find $Z_T$:

$$Z_T = \frac{Z_{R1} \times Z_L}{(Z_{R1} + Z_L)}$$

$$= \frac{(2) \times (j2)}{(2 + j2)}$$

$$= 1 + j1.$$
Thevenin circuit becomes:

\[ 1 + j1 \]

\[ 6 + j18 \]

Reconnect C and R₀:

\[ 1 + j1 \]

\[ 6 + j18 \]

\[ + \]

\[ - \]

\[ C \]

\[ R₀ \]

\[ + \]

\[ V₀ \]

\[ - \]

Then

\[ V₀ = V_T Z_{R₀} / (Z_{R₀} + Z_C + Z_T) \]

\[ = (6 + j18) (2) / (2 – j2 + (1 + j1)) \]

\[ = 12 (1 + j3) / (3 – j1) \]

\[ = 12 (3 + j9 + j -3) / 10 \]

\[ = j12 \]

\[ = 12 \angle 90 \text{ deg} \]

END EXAMPLE 2.5.

Example 2.6. Thevenin equivalent with dependent source:

Dependent voltage source \( V_{dep} \) where

\[ V_{dep} = 2 \, I_x. \]

Find \( V_{OC}, Z_T \) using Thevenin’s theorem.

\( V_{OC}: \)

\[ I_x = V_S / (Z_L + Z_C) \]

\[ = (24 \angle 0) / (j2 – j1) \]

\[ = -j24 \, A \]

\[ V_{OC} = Z_C \, I_x - 2 \, I_x \]

\[ = (-j1) (-j24) - 2 (-j24) \]

\[ = -24 + j48. \]

Find \( I_{SC}: \)

2 mesh equations:

(1) \[ V_S = I_x (j2) + (I_x - I_{SC}) \, (-j1) = I_x \, (j1) + I_{SC} \, (j1) \]

(2) \[ 0 = (I_{SC} - I_x) \, (-j1) + 2 \, I_x = I_x \, (2 + j1) - I_{SC} \, (j1) \]

From (2)

\[ I_x = I_{SC} \, (j1) / (2 + j1) \]

Substitute in (1):

\[ V_S = I_{SC} \, (j1) \, (j1) / (2 + j1) + I_{SC} \, (j1) \]

\[ = I_{SC} \, [-1] / (2 + j1) + (j1)] \]

\[ = I_{SC} \, [-1] / (2 + j1) + (j1) (2 + j1) / (2 + j1)] \]

\[ = I_{SC} \, (-1 + j2 - 1) / (2 + j1) \]

\[ = I_{SC} \, (-2 + j2) / (2 + j1) \]

So

\[ I_{SC} = V_S \, (2 + j1) / (-2 + j2) \]

\[ = (24 \angle 0) \, (2 + j1) / (-2 + j2) \]

\[ = 24 \, (2 + j1) \, (-2 - j2) / 8 \]

Let \( V_S = 24 \angle 0 \text{ deg} \)

\( Z_L = j2, \)

\( Z_C = -j1 \)
Then can replace current source and parallel capacitance with voltage source and series capacitance:

\[ V_S \]

\[ + \]

\[ 2 \]

\[ - \]

\[ j_2 \]

\[ V_O \]

Note that \( Z_T = Z_N = -j2 \).

Find \( V_S = V_T = V_{OC} \):

\[ V_{OC} = Z_N I_N \]

\[ = (5 \angle 0) (-j2) \]

\[ = -j10 \]

\[ = 10 \angle -90 \text{ deg V.} \]

Then,

\[ V_O = V_S (2) / (-j2 + j2 + 2) \]

\[ = V_S (2) / (2) \]

\[ = V_S \]

\[ = 10 \angle -90 \text{ deg V.} \]

END EXAMPLE 2.7.

**Superposition.**

With multiple sources, can find solution for each source, letting other sources = 0 (voltage sources shorted, current sources open). Then sum all individual solutions.
Example 2.8. Superposition.

Let $V_1 = 5 \angle 0$ deg V,
$I_2 = 10 \angle 90$ deg A,
$Z_{R1} = 2$,
$Z_{R2} = 4$,
$Z_C = -j4$.

Find $V_O$.
Solution 1. Open current source ($I_2 = 0$), solve with only voltage source:

Find impedance of $C$ and $R_2$: $Z_L = (4)(-j4)/(4-j4) = 2-j2$. Then
$V_O = (5\angle0)(2-j2)/(2-j2+2) = 5(1-j1)/(2-j1)\angle-45 = 3.16\angle-18.44$ deg V.

Solution 2. Short voltage source ($V_1 = 0$), solve with only current source:

Combine solutions:
$V_O = V_{O1} + V_{O2} = (3.16 \angle -18.44) + (12.65 \angle 71.56) = 3.00 - j1.00 + 4.00 + j12.00 = 7.00 + j11.00 = 13.04 \angle 57.53$ deg V.

END EXAMPLE 2.8.

Example 2.9 Sources at different frequencies:
Superposition must be used when sources are at different frequencies. Reactances are frequency dependent, so will change.
v₁(t) = 24 cos (1000t)
i₂(t) = 4 cos (2000t + 45 deg)
R₁ = 2
C = 500 uF

Solution 1: i₂ = 0: V₁ = 24 ∠ 0, ω = 1000 rad/sec
Z₈ = 1/j(1000) (500u) = -j2
So
V₀ = V₁ Z₈ / (Z₈ + R)
   = (24 ∠ 0) (-j2) / (2 – j2)
   = (48 ∠ -90) / (2 sqrt(2) ∠ -45)
   = 16.97 ∠ (-90 – (-45))
   = 16.97 ∠ - 45 deg V.

or
V₀(t) = 16.97 cos (1000t - 45 deg) V

Solution 2: v₁ = 0: I₂ = 4 ∠ 45 deg, ω = 2000 rad/sec
Z₈ = 1/j(2000) (500u) = -j1
So
V₀ = I₂ Ztot
   = I₂ Z₈ Z₉ / (Z₈ + Z₉)
   = (4 ∠ 45) (2) (-j1) / (2 - j1)
   = (8 ∠ -45) / (sqrt(5) ∠ -26.57)
   = 3.58 ∠ - 18.43 deg V

or
V₀(t) = 3.58 cos (2000t – 18.43 deg) V

So,
V₀(t) = V₀₁(t) + V₀₂(t) = 16.97 cos (1000t - 45 deg) V + 3.58 cos (2000t – 18.43 deg) V

Cannot write solution as single phasor because sources at different frequencies.
END EXAMPLE 2.9.

Ideal Op Amps
Operational amplifier is analyzed as an ideal circuit with infinite gain and infinite input impedance. The (+) signifies the non-inverting input, (-) is inverting input. All voltages are referenced to ground.

An ideal op amp obeys 2 rules:

1. Infinite input impedance: I₊ = I₋ = 0.

2. Infinite gain: V₊ - V₋ = 0.

Example 2.10. Ideal Op Amp.

I₁ = -I₂ (since I₋ = 0)

And
I₁ = Vₛ / Z₁, (since V₊ = V₋ = 0)
And
\[ I_2 = \frac{V_O}{Z_2}, \]
Combining,
\[ V_O = -\left(\frac{Z_2}{Z_1}\right) V_S. \]

So if \( Z_1 = R \) and \( Z_2 = \frac{1}{j\omega C} \), then
\[ V_O = -\left(\frac{1}{j\omega RC}\right) V_S. \]
\[ = \left(\frac{j}{\omega RC}\right) V_S. \]
This is low-pass filter, since
\[ V_O \rightarrow \infty \text{ as } \omega \rightarrow 0, \]
And
\[ V_O \rightarrow 0 \text{ as } \omega \rightarrow \infty, \]
Note that output is shifted by 90 deg from input.

END EXAMPLE 2.10

End Chapter 2

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