More on Incomplete Gauss Elimination (IGE), and LU Matrix Factorization:

Ex: Consider:

\[
A = \begin{bmatrix}
3 & 2 & -1 \\
-6 & 2 & 2 \\
3 & -1 & -2
\end{bmatrix} \quad b = \begin{bmatrix}
4 \\
8 \\
9
\end{bmatrix}
\]

Then, the matrix that describes the IGE pivot in the first column (multiply first row by 2 and add to second row; multiply first row by -1 and add to third row) is

\[
E_1 = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]

\[
E_1A = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 2 & -1 \\
-6 & 2 & 2 \\
3 & -1 & -2
\end{bmatrix} = \begin{bmatrix}
4 \\
16 \\
5
\end{bmatrix}
\]

and, the matrix that describes the IGE pivot in the second column (multiply second row by \(\frac{1}{2}\) and add to third row) is

\[
E_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} & 1
\end{bmatrix}
\]

\[
E_2E_1A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} & 1
\end{bmatrix}
\begin{bmatrix}
3 & 2 & -1 \\
0 & 6 & 0 \\
0 & 0 & -1
\end{bmatrix} = \bar{A} \quad E_2E_1b = \begin{bmatrix}
4 \\
16 \\
13
\end{bmatrix}
\]

Of course, at this point the problem is solved.

By backward substitution, we get

\[
\bar{x}_3 = -13 \\
\bar{x}_2 = \frac{16}{6} = \frac{8}{3} \\
\bar{x}_1 = \frac{(4-2\bar{x}_2+\bar{x}_3)}{3} = \frac{-43}{9}
\]

But this is not the only interesting result. We also have

\[
E_2E_1A \equiv \bar{A} = U
\]

an "upper triangular" matrix, and also \(E_2\) and \(E_1\) are invertible

Therefore, \(A = (E_2E_1)^{-1} \bar{A} = LU\) where \(L = E_1^{-1}E_2^{-1}\) and \(U = \bar{A}\). In fact, in this example
\[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
1 & -\frac{1}{2} & 1
\end{bmatrix} = L
\]

Therefore in this case, we have \( A = (E_2E_1)^{-1} \vec{A} \equiv LU \), and,

\[
\det A = \det LU = \det L \cdot \det U = 1 \cdot (-18) = -18
\]

However, recall the example (in \((\beta, n) = (10, 4)\) arithmetic)

\[
\begin{align*}
0.0003x_1 + 1.566x_2 &= 1.569 \\
0.3454x_1 - 2.436x_2 &= 1.018
\end{align*}
\]

The exact solution is \( x^* = (10,1)^T \). But, in \((10, 4)\) arithmetic, applying Incomplete Gauss Elimination, we get:

\[
E_1 = \begin{bmatrix}
1 & 0 \\
-0.3454 & 1 \\
0.0003 & 1
\end{bmatrix}^{(10,4)} = \begin{bmatrix}
1 & 0 \\
-1151 & 1
\end{bmatrix}
\]

Now \( a_{22} = -2.436 \), but

\[
\begin{align*}
\bar{a}_{22} &= -2.436 + 1.566 \cdot \left(\frac{-0.3454}{0.0003}\right) \\
&= -2.436 - 1.566 \cdot 1151 \\
&= -2.436 - 1802 = -1804 \\
\bar{b}_2 &= 1.018 - 1.569 \cdot 1151 \\
&= 1.018 - 1806 = 1805
\end{align*}
\]

So, we get \( \bar{x}_2 = \frac{-1805}{-1804} = 1.001 \) and therefore

\[
\bar{x}_1 = \frac{(1.569 - 1.566 \bar{x}_2)}{0.0003} = \frac{(1.569 - 1.566 \cdot 1.001)}{0.0003} = \frac{(1.569 - 1.568)}{0.0003} = 3.333
\]

Hence, we have that \( \bar{x} = (3.333, 1.001)^T \) which is nowhere near the exact answer of \( x^* = (10,1)^T \).

This leads to Incomplete Gauss elimination with partial pivoting (i.e. Gauss elimination with row interchanges), which we'll denote by IGEPP.

**The idea:** The pivot element should be the largest in absolute value. Let's go back to the example above and use IGEPP.
Therefore, first we will interchange rows 1 and 2 using the permutation matrix $P_1$,

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1A = \begin{bmatrix} -6 & 2 & 2 \\ 3 & 2 & -1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P_1b = \begin{bmatrix} 8 \\ 4 \\ 9 \end{bmatrix}$$

This gives us

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$E_1P_1A = \begin{bmatrix} -6 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E_1P_1b = \begin{bmatrix} 8 \\ 8 \\ 13 \end{bmatrix}$$

Therefore, we're finished since we've arrived at an upper triangular matrix. Or, if one wishes, we can think of the IGEPP pivot operation in the second column as the "do nothing" operation. This, of course, would be represented by the identity matrix. That is

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have an LU factorization of the matrix $P_1A$. That is, we have

$$P_1A = E_1^{-1}E_2P_1A = E_1^{-1}E_2 \begin{bmatrix} -6 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = LU$$

We also have that

$$\det P_1A = \det L \cdot \det U$$

$$= 1 \cdot 18$$

$$\det A = -18$$
Ex: Consider

\[ A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 9 \\ 3 & 6 & 9 \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ 16 \\ 18 \end{bmatrix} \]

The exact solution is given by \( x^* = (1, 1, 1)^T \). First we will interchange rows 1 and 3 using the permutation matrix \( P_1 \).

\[ P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ P_1A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 9 \\ 1 & 4 & 7 \end{bmatrix} \quad P_1b = \begin{bmatrix} 18 \\ 16 \\ 12 \end{bmatrix} \]

The elementary matrix that represents the IGE pivot on "3" in the first column is

\[ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \]

\[ E_1P_1A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad E_1P_1b = \begin{bmatrix} 18 \\ 6 \\ 4 \end{bmatrix} \]

Next, we interchange rows 2 and 3 using the permutation matrix \( P_2 \).

\[ P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ P_2E_1P_1A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad P_2E_1P_1b = \begin{bmatrix} 18 \\ 6 \\ 4 \end{bmatrix} \]

The next elementary matrix \( E_2 \), that represents the IGE pivot in the second column is

\[ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \]

\[ E_2P_2E_1P_1A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2P_2E_1P_1b = \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} \]

Therefore, we have an “LU” factorization of the matrix where \( E_2P_2E_1P_1A = \tilde{A} = U \). We have
\[
P_A = (E_2 E_1)^{-1} \bar{A}
\]
\[
P_2 P_A = P_2 (E_2 E_1)^{-1} \bar{A}
\]

**Note:** \( P_2 \) is neither “U” nor “L”

However, suppose we proceed as if we knew in advance all row interchanges that would be required for this example. In other words, for our example we know that we will first interchange rows 1 and 3 and then interchange rows 2 and 3.

\[
P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

We would then get

\[
P_1 A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 9 \\ 1 & 4 & 7 \end{bmatrix} \quad P_2 b = \begin{bmatrix} 18 \\ 16 \\ 12 \end{bmatrix}
\]

\[
P_2 P_1 A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix} \quad P_2 P_1 b = \begin{bmatrix} 12 \\ 16 \end{bmatrix}
\]

The elementary lower triangular matrices resulting are

\[
\hat{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix}
\]

\[
\hat{E}_1 P_1 P_2 A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad \hat{E}_1 P_2 P_1 b = \begin{bmatrix} 18 \\ 6 \\ 4 \end{bmatrix}
\]

\[
\hat{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}
\]

\[
\hat{E}_2 \hat{E}_1 P_1 P_2 A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{E}_2 \hat{E}_1 P_2 P_1 b = \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}
\]

Therefore we see that

\[
P_2 P_1 A = (\hat{E}_2 \hat{E}_1)^{-1} \bar{A} \equiv LU
\]

where
\[ U = \bar{A} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ L = \left( \hat{E}_2 \hat{E}_1 \right)^{-1} = \hat{E}_1^{-1} \hat{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix} \]

That is, we have factored \( P_2 P_1 A \) into the two L and U matrices above.

**Exercise:** Show that this is the same as the previous \( P_2 \left( E_2 P_2 E_1 \right)^{-1} \)

**The Doolittle Method for Computing an LU Factorization:**

Now, we have seen that by applying Incomplete Gauss Elimination we get an LU factorization if it exists. Also note that Incomplete Gauss Elimination always produces an L with “ones” on the main diagonal. This leads to the following question. Is there another way to factor A into the product of two matrices, L and U, where L is lower triangular and U is upper triangular?

The answer is yes and the method is known as Doolittle's method.

**Ex:** Solve for L and U given the matrix A and the relation A=LU

\[ \begin{bmatrix} 3 & 2 & -1 \\ -6 & 2 & 2 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \]

If there are two such matrices, then it must be the case that the following holds. Let \( l_i \) denote the ith row of L, and let \( u_j^i \) denote the jth column of U. Then matrix multiplication implies the following for the first row of U.

\[ 3 = l_1 u_1^1 = u_{11} \]
\[ 2 = l_1 u_2^1 = u_{12} \]
\[ -1 = l_1 u_3^1 = u_{13} \]

Therefore, if there are two such matrices, the first row of U must be comprised of these numbers. We'll now use this information to determine the first column of L.
Once the first row of \(U\) is determined, we can determine the first column of \(L\)

\[-6 = l_2u^1 \Rightarrow l_{21} = -2\]

\[3 = l_3u^1 \Rightarrow l_{31} = -1\]

That is, if there are two such matrices, the first column of \(L\) must be comprised of these numbers. We'll now use this information to determine the second row of \(U\). The matrix equation is now

\[
\begin{bmatrix}
3 & 2 & -1 \\
-6 & 2 & 2 \\
3 & -1 & -2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
l_{21} & 1 & 0 \\
l_{31} & l_{32} & 1
\end{bmatrix}
\begin{bmatrix}
3 & 2 & -1 \\
u_{22} & u_{23} \\
u_{32} & u_{33}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & u_{33}
\end{bmatrix}
\]

\[2 = l_2u^2 = -4 + u_{22} \Rightarrow u_{22} = 6\]

\[2 = l_2u^3 = 2 + u_{23} \Rightarrow u_{23} = 0\]

Therefore, we now have

\[
\begin{bmatrix}
3 & 2 & -1 \\
-6 & 2 & 2 \\
3 & -1 & -2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
l_{21} & 1 & 0 \\
l_{31} & l_{32} & 1
\end{bmatrix}
\begin{bmatrix}
3 & 2 & -1 \\
u_{22} & u_{23} \\
u_{32} & u_{33}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & u_{33}
\end{bmatrix}
\]

Now, to determine the second column of \(L\)

\[-1 = l_3u^2 = 2 + 6l_{32} \Rightarrow l_{32} = -\frac{1}{2}\]

Therefore

\[
\begin{bmatrix}
3 & 2 & -1 \\
-6 & 2 & 2 \\
3 & -1 & -2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
l_{21} & 1 & 0 \\
l_{31} & l_{32} & 1
\end{bmatrix}
\begin{bmatrix}
3 & 2 & -1 \\
u_{22} & u_{23} \\
u_{32} & u_{33}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & u_{33}
\end{bmatrix}
\]

And finally

\[-2 = l_3u^3 = -1 + u_{33} \Rightarrow u_{33} = -1\]

We see that the Doolittle method has succeeded and we've found an LU factorization for the matrix \(A\). Note that it's precisely the same as the one we found by IGE.

Recall the advantage of an LU factorization. Given:
\[ \begin{align*}
Ax &= b \\
LUx &= b \quad \text{Let } y = Ux \\
Ly &= b
\end{align*} \]

We have the following set of calculations that are easily executed via the so-called process of forward substitution:

\[ \begin{align*}
\bar{y}_1 &= \frac{b_1}{a_{11}} \\
\bar{y}_2 &= \frac{b_2 - b_{11}\bar{y}_1}{a_{22}} \\
\bar{y}_3 &= \frac{b_3 - b_{11}\bar{y}_1 - b_{21}\bar{y}_2}{a_{33}} \\
&\vdots \\
\bar{y}_n &= \frac{b_n - b_{11}\bar{y}_1 - \cdots - b_{n-1,1}\bar{y}_{n-1}}{a_{nn}}
\end{align*} \]

Given \( \bar{y} \), we now solve

\[ \begin{align*} 
Ux &= \bar{y} 
\end{align*} \]

We then have the following set of calculations that are easily executed via the so-called process of back substitution:

\[ \begin{align*}
\bar{x}_n &= \frac{\bar{y}_n}{u_{nn}} \\
\bar{x}_{n-1} &= (\bar{y}_{n-1} - u_{n-1,n}\bar{x}_n) / u_{n-1,n-1} \\
&\vdots 
\end{align*} \]

**Ex ( a singular matrix with an LU factorization):**

\[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

Let \( y = Ux \)

\[ \begin{align*}
Ly &= LUx = b \\
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\Rightarrow \bar{y}_1 &= 1 \\
\Rightarrow \bar{y}_2 &= 0
\end{align*} \]

This gives us

\[ \begin{align*}
Ux &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{align*} \]

This means that \( \bar{x}_2 = \text{anything} \) and \( \bar{x}_1 = 1 - \bar{x}_2 = \text{anything} \). Our system of equations has many solutions.