Lecture 9

- System defined by a differential equation.
- Transfer function of an LTI, causal system.
- Cascaded systems and other block diagram interconnections.

System defined by a differential equation

Assume all signals are zero for $t < 0$. For $t \geq 0$, the input and output are related by

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m x}{dt^m} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

Using Laplace (zero initial conditions), we get

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0)Y(s) = (b_ms^n + \cdots + b_1s + b_0)X(s)$$

$$Y(s) = \frac{(b_ms^n + \cdots + b_1s + b_0)}{(s^n + \cdots + a_1s + a_0)}X(s) = H(s)X(s)$$
In the Laplace domain, the input and output are related by a simple multiplication by a certain function $H(s)$. This is a first example of a **transfer function** of an LTI system.

Example: mass-spring system

$$m \frac{d^2 z}{dt^2} = f(t) - kz \rightarrow m \frac{d^2 z}{dt^2} + kz = f(t)$$

$$\left(ms^2 + k\right)Z(s) = F(s) \rightarrow Z(s) = \frac{1}{\left(ms^2 + k\right)}F(s)$$

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**Transfer function of an LTI, causal system**

Recall: $y(t) = h(t) \ast x(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$

Theorem: Given an LTI, causal system, with impulse response $h(t)$; suppose the input satisfies $x(t) = 0$ for $t < 0$. Let $X(s) = \mathcal{L}[x(t)]$, $H(s) = \mathcal{L}[h(t)]$, $Y(s) = \mathcal{L}[y(t)]$. Then

$$Y(s) = H(s)X(s)$$

for any $s$ in the DOC of both $H(s)$ and $X(s)$.
\textbf{Proof:}

Using causality, \( h(t) = 0 \) for \( t < 0 \). Therefore:

\[ y(t) = \int_{-\infty}^{\infty} h(\sigma)x(t-\sigma) d\sigma = \int_{0^-}^{\infty} h(\sigma)x(t-\sigma) d\sigma \]

\[ Y(s) = \int_{0^-}^{\infty} e^{-st}y(t) dt = \int_{0^-}^{\infty} e^{-st}\left[\int_{0^-}^{\infty} h(\sigma)x(t-\sigma) d\sigma\right] dt \]

\[ = \int_{0^-}^{\infty} h(\sigma)\left[\int_{0^-}^{\infty} e^{-st}x(t-\sigma) d\sigma\right] d\sigma \]

\[ \mathcal{L}[u(t-\sigma)x(t-\sigma)] \]

\[ = \int_{0^-}^{\infty} h(\sigma)e^{-sx}X(s) d\sigma = H(s)X(s). \]

\[ \text{Note: by hypothesis, } x(t) = u(t)x(t). \]

\[ (x(t) = 0 \text{ for } t < 0). \]

\[ \text{Delay property} \]

\[ \text{Example:} \]

\[ h(t) = \delta(t) - u(t)e^{-t}; \quad x(t) = t \ u(t). \quad \text{Find } y(t). \]

\[ H(s) = 1 - \frac{1}{s+1} = \frac{s}{s+1}; \quad X(s) = \frac{1}{s^2}. \]

\[ Y(s) = H(s)X(s) = \frac{s}{s+1} \cdot \frac{1}{s^2} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \]

\[ \Rightarrow y(t) = u(t)\left[1 - e^{-t}\right]. \]

\[ \text{Easier than convolution!} \]
Remarks:

- As defined in this course, the Laplace transform only considers times $t \geq 0$. "One-sided" transform.

- As such, it is used to study signals and LTI systems which start operating at a given time (for convenience, chosen to be 0).

- If a problem involves signals starting at $t = -\infty$, or non-causal systems, we do not apply the previous result with Laplace. At this point in the course, we can only approach it in the time domain, via convolutions.

- Later on, we will introduce Fourier transforms that can be used to study signals involving negative times.

Definition: The Laplace transform $H(s) = \mathcal{L}[h(t)]$ of the system impulse response function is called the transfer function (or system function) of the LTI, causal system.

Transfer function of a cascaded system

\[
\begin{align*}
z &= h_2 \ast y = h_2 \ast h_1 \ast x \\
\rightarrow Z(s) &= H_2(s)Y(s) = \frac{H_2(s)H_1(s)X(s)}{H_{12}(s)}
\end{align*}
\]
More generally, we can build “block-diagrams”

\[ x \rightarrow H_1(s) \rightarrow y_1 \rightarrow H_3(s) \rightarrow z \]

\[ y_2 \]

Notation:
\( H(s) \) inside a box means an LTI system with this transfer function. The "adder" block \( \oplus \) represents \( y(t) = y_1(t) + y_2(t) \).

\[ Z(s) = H_3(s)Y(s) = H_3(s)[Y_1(s) + Y_2(s)] \]
\[ = H_3(s)[H_1(s)X(s) + H_2(s)X(s)] \]
\[ = H_3(s)[H_1(s) + H_2(s)]X(s). \]

\( H(s) \), overall transfer function.

Feedback interconnection

\[ x \rightarrow y \rightarrow H_1(s) \rightarrow z \]

\[ y \]

\[ H_2(s) \]

\[ Z(s) = H_1(s)Y(s) \]
\[ Y(s) = X(s) + H_2(s)Z(s) \]
\[ \rightarrow Z(s) = H_1(s)X(s) + H_1(s)H_2(s)Z(s) \]
\[ \rightarrow [1 - H_1(s)H_2(s)]Z(s) = H_1(s)X(s) \]
\[ Z(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}X(s). \]

Main lesson:
Use simple algebra to study complex systems.
Example: build a transfer function from simple blocks.

Integrator block: \( y(t) = \int_{0^-}^{t} x(\tau) d\tau \)

In Laplace, \( Y(s) = \frac{1}{s} X(s) \)

Amplifier: \( y(t) = a x(t) \)

In Laplace, \( Y(s) = a X(s) \)

There exist circuits (e.g., based on OP-AMPs) that approximately implement these basic functions.

Now, we use them to build a more complicated transfer function. An “analog computer”.

\[
Y(s) = a_0 X(s) + a_1 \frac{1}{s} X(s) + a_2 \frac{1}{s^2} X(s) + \cdots + a_n \frac{1}{s^n} X(s)
\]

\[
= \left( a_0 + \frac{a_1}{s} + \cdots + \frac{a_n}{s^n} \right) X(s) = \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}{s^n} X(s)
\]

We can build any numerator polynomial by choosing \( a_0, \ldots, a_n \).

Also, using feedback one can build different denominators.