Lecture 5: Convolutions and Applications

Recall: Input-output relation of a linear system

\[ x \rightarrow y \quad y(t) = T[x(t)] \]

Let the impulse response function be \( h(t, \tau) = T[\delta(t - \tau)] \).

For a given input \( x(t) \), the corresponding output is

\[ y(t) = T[x(t)] = \int_{-\infty}^{\infty} h(t, \sigma) x(\sigma) d\sigma \]

SUPERPOSITION INTEGRAL

For the time invariant case: \( h(t, \tau) = h(t - \tau) \)

\[ y(t) = \int_{-\infty}^{\infty} h(t - \sigma) x(\sigma) d\sigma \]

This operation is called the convolution of the functions \( h(t) \) and \( x(t) \). Notation:

\[ y(t) = h(t) * x(t) \quad \text{or} \quad y = h * x \]

Example: Given \( h(t) \), \( x(t) \) below, find \( y(t) = h(t) * x(t) \).

Given two functions of one variable, the convolution operation returns another function of one variable.
\[ y(t) = \int_{-\infty}^{t} h(t-\sigma)x(\sigma)d\sigma \]

- \(0 \leq t < 1\)

- \(1 \leq t < 2\)

- \(2 \leq t < 3\)

- \(3 \leq t\)
Properties of convolution. 
\[ (f \ast g)(t) = \int_{-\infty}^{\infty} f(t - \sigma)g(\sigma)d\sigma \]

1) Associative: \( (f \ast g) \ast h = f \ast (g \ast h) \)
2) Commutative: \( f \ast g = g \ast f \)
3) Distributive: \( f \ast (g + h) = f \ast g + f \ast h \)
4) Unit of convolution: \( f \ast \delta = f \)

Algebraic properties of a product!
Impulse response of a cascaded system

Consider two LTI systems, S1 and S2.
S1 has input $x(t)$, output $y(t)$, and impulse response $h_1(t)$.
S2 has input $y(t)$, output $z(t)$, and impulse response $h_2(t)$.

The cascade has input $x(t)$, output $z(t)$. It is easy to see, using the definitions, that it is also an LTI system.
We wish to find its impulse response $h_{1,2}(t)$.

Impulse response of a cascaded system

Applying an input $x(t) = \delta(t)$ to the cascade
By definition, $y(t) = h_1(t)$ (impulse response of S1)
Then for S2 we have $z = h_2 \ast y = h_2 \ast h_1$

Conclusion: The impulse response of the cascade is the convolution of the impulse responses of each stage.

$h_{1,2} = h_2 \ast h_1$

Associativity of convolution

S1,S2 LTI. We have seen that $h_{1,2} = h_2 \ast h_1$
Applying now a generic input $x(t)$, we have
$z = (h_2 \ast h_1) \ast x$.
Looking at S1 with that input, we have $y = h_1 \ast x$
Now S2 gives $z = h_2 \ast y = h_2 \ast (h_1 \ast x)$.

Conclusion: $(h_2 \ast h_1) \ast x = h_2 \ast (h_1 \ast x)$.

A consequence of commutativity

Since $h_2 \ast h_1 = h_1 \ast h_2$, the above cascades of LTI systems have the same impulse response.
Therefore, they are equivalent: LTI systems commute.
Note: they are equivalent only as mappings from $x$ to $z$.
The intermediate signals $y$ and $v$ will not be the same.
A cascaded circuit

Recall: \( h(t) = \alpha e^{-\alpha t} u(t) \)
\( \alpha = \frac{1}{RC} \)

Now consider:

True or false: is the impulse response of this circuit equal to \( h(t) = \left[ \alpha_1 e^{-\alpha_1 t} u(t) \right] * \left[ \alpha_2 e^{-\alpha_2 t} u(t) \right] \)?

Answer: False!