Systems and Signals

Lecture 4: Systems Characteristics

October 5, 2011
Agenda

• My office hours is now T 2-4pm.

• Zhongnan’s office hour is now T 11-12pm.

• Discussion C (T 2-3pm) and D (T 3-4pm) will take place at Boelter Hall 5272, starting from this week.
System Characteristics

Today's topics:

• System interconnections
• Characteristics (memory, causality, stability)
• Differential equations
A signal can be of one of the following types:

- **Energy signal,**

\[ 0 < E_x = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt < \infty \]

- **Power signal,**

\[ 0 < P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt < \infty \]

- Neither *energy* nor *power* signal. For instance, \( E_x = \infty \) and \( P_x = 0 \).

A signal CANNOT be both an *energy* and *power* signal. Why?
• An energy signal $x(t)$ has zero power

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = 0$$

$\rightarrow E_x < \infty$

• A power signal has infinite energy

$$E_x = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} 2T \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \infty.$$  

$\rightarrow P_x > 0$

• But, a signal with zero power may or may not be an energy signal.

• Similarly, a signal with infinite energy may or may not be a power signal.
System Characteristics

- A system transforms *input signals* into *output signals*.
- A system is a *function* mapping input signals into output signals.
- Systems often denoted by *block diagrams*.
Linearity

• A system is linear if it satisfies homogeneity and additivity properties.

• Homogeneity: \( F(ax) = aF(x) \)

• Additivity: \( F(x + \tilde{x}) = F(x) + F(\tilde{x}) \)

• Another useful test: zero input to a linear system must produce zero output (ZIZO property).
  If \( x(t) = 0 \), and \( F(x) \neq 0 \), the system \( F \) is not linear!
Combined Homogeneity and Superposition:

- If $y_1 = Sx_1$ and $y_2 = Sx_2$, and $a$ and $b$ are constants,

$$ay_1 + by_2 = S(ax_1 + bx_2)$$

Extended Linearity

Given system $S$, input signals $x_n$, output signals $y_n$, scalars (constants) $a_n$.

- Summation: If $y_n = Sx_n$ for all $n$, an integer from $(-\infty < n < \infty)$

$$\sum_n a_n y_n = S \left( \sum_n a_n x_n \right)$$

Summation and the system operator commute, and can be interchanged.
Consider the circuit (a system) above. 
Input signal is voltage $V(t)$. 
System output is current through the resistor $I_R(t)$. 
Ohm’s Law: $I_R(t) = V(t)/R$. 

Is the system linear?
Consider a circuit with the resistor replaced by the diode. Input signal is voltage $V(t)$. System output is current through the diode $I_D(t)$. Current through the diode is given by

$$I_D(t) = I_s(e^{V(t)/V_T} - 1)$$

where $I_s$ and $V_T$ are physical parameters not relevant to the discussion.

Is the system linear?
Time Invariance

- Conceptually: system response is independent of time.

- A system is time-invariant if a time shift in the input only produces the same time shift in the output.

- For a system $F$,

$$y(t) = Fx(t)$$

implies that

$$y(t - \tau) = Fx(t - \tau)$$

for any time shift $\tau$ and input signal $x$. 
Consider the system $F$ defined by the input-output relationship

$$y(t) = \sin(x(t))$$

Is the system time invariant?

Check system output if the \textit{input} is delayed:
Delay input: $\hat{x}(t) = x(t - T)$
Apply system: $F(\hat{x}) = \sin(\hat{x}(t)) = \sin(x(t - T))$

Check system output if the \textit{output} is delayed:
Apply system: $F(x) = \sin(x(t)) = y(t)$
Delay output: $\hat{y}(t) = y(t - T) = \sin(x(t - T))$

\textit{System is time-invariant.}
Continuous Time Unit Impulse (Dirac delta)

Compute

\[ y(t) = \int_{-1}^{\infty} \delta(t - \tau)e^{-2\tau} d\tau, \quad -\infty < t < \infty \]
Let consider $\delta(t - \tau)$ as a function of $\tau$. Thus, $\delta(t - \tau)$ is an impulse at $t$ (i.e., $\delta(t - \tau) = \infty$ at $\tau = t$). We have two cases:

- $-\infty < t < -1$
  In this case, the impulse locates outside the range of integration. Hence
  
  $$y(t) = \int_{-1}^{\infty} \delta(t - \tau)e^{-2\tau} d\tau = 0$$

- $-1 \leq t < \infty$
  In this case, the impulse locates inside the range of integration. Hence
  
  $$y(t) = \int_{-1}^{\infty} \delta(t - \tau)e^{-2\tau} d\tau = e^{-2t}$$

We find that

$$y(t) = \int_{-1}^{\infty} \delta(t - \tau)e^{-2\tau} d\tau = \begin{cases} 0, & -\infty < t < -1 \\ e^{-2t}, & -1 \leq t < \infty \end{cases}$$
Conclusion: The integral of the product of a function and an impulse function is the value of the function computed at the location of the impulse.
Unit Step Function

Compute

\[ \int_{-\infty}^{\infty} u(t - \tau)u(\tau - \sigma)d\tau, \quad -\infty < t, \sigma < \infty. \]

*Hint:* Plot the product \( u(t - \tau)u(\tau - \sigma) \) as a function of \( \tau \)
\[ u(t - \tau)u(\tau - \sigma) = \begin{cases} 1, & \sigma \leq \tau \leq t \\ 0, & \text{otherwise.} \end{cases} \]

Therefore,

\[
\int_{-\infty}^{\infty} u(t - \tau)u(\tau - \sigma)d\tau = \begin{cases} \int_{\sigma}^{t} 1d\tau, & t \geq \sigma \\ 0, & t < \sigma \end{cases} = \begin{cases} t - \sigma, & t \geq \sigma \\ 0, & t < \sigma \end{cases}
\]

\[= (t - \sigma)u(t - \sigma), \quad -\infty < t, \sigma < \infty \]
Interconnection of Systems

We can interconnect systems to form new systems,

- **cascade (or series):** \( y = G(F(x)) = GFx \)

![Diagram of cascade connection]

(note that block diagrams and algebra are reversed)

- **sum (or parallel):** \( y = Fx + Gx \)

![Diagram of sum connection]
• **feedback:** \( y = F(x - Gy) \)

In general,

• Block diagrams are a symbolic way to describe a connection of systems.
• We can just as well write out the equations relating the signals.
• We can go back and forth between the system block diagram and the system equations.
Example: Integrator with feedback

Input to integrator is $x - ay$, so

$$\int_0^t (x(\tau) - ay(\tau)) \, d\tau = y(t)$$

Another useful method: the *input* to an integrator is the derivative of its output, so we have

$$x - ay = y'$$

(of course, same as above)
System Memory

- A system is *memoryless* if the output depends only on the present input.
  - Ideal amplifier
  - Ideal gear, transmission, or lever in a mechanical system

- A *system with memory* has an output signal that depends on inputs in the past or future.
  - Energy storage circuit elements such as capacitors and inductors
  - Springs or moving masses in mechanical systems

- A *causal* system has an output that depends only on past or present inputs.
  - Notice that any memoryless system must also be causal.
  - Any real physical circuit, or mechanical system.
Are these systems memoryless? Are they causal?

- $y(t) = x(-t^2)$
- $y[n] = x[n - 1]$
- $y[n] = x[n^2]\delta[n]$
- $y(t) = x(-t)$
• \( y(t) = x(-t^2) \)
• \( y(t) = x(-t^2) \)

This system is \textit{not memoryless} (when \( t > 0 \)), and it is \textit{non-causal} because \( t < -t^2 \) for \(-1 < t < 0\).
\bullet y[n] = x[n - 1]
\[ y[n] = x[n - 1] \]

This system is \textit{not memoryless}, but it is \textit{causal}.
\[ y[n] = x[n^2] \delta[n] \]
\[ y[n] = x[n^2] \delta[n] \]

From lecture 3, \( y[n] = x[n^2] \delta[n] = x[0] \delta[n] \).
For \( n \neq 0 \), \( y[n] = 0 \) and is independent from \( x[n] \).
When \( n = 0 \), \( y[n] \) depends on the present input \( x[0] \).

Hence, this is a memoryless and causal system.
\[ y(t) = x(-t) \]
\[ y(t) = x(-t) \]

When \( t < 0 \), the output depends on the future input. When \( t > 0 \), the output depends on the past input.

Therefore, this system is neither memoryless nor causal.
System Stability

- Stability important for most engineering applications.
- Many definitions
- If a bounded input
  \[ |x(t)| \leq M_x < \infty \]
  always results in a bounded output
  \[ |y(t)| \leq M_y < \infty, \]
  where \( M_x \) and \( M_y \) are finite positive numbers, the system is *Bounded Input Bounded Output (BIBO) stable*. 
Are these systems BIBO stable?

• Accumulator system

\[ y[n] = \sum_{i=-\infty}^{n} x[i] \]

• Discrete time echo

\[ y[n] = \alpha y[n-1] + x[n] \]

assuming \( y[n] = 0, \forall n < 0 \), so that \( y[0] = x[0] \).

See that \( y[1] = \alpha x[0] + x[1] \), \( y[2] = \alpha^2 x[0] + \alpha x[1] + x[2] \), and in general

\[ y[n] = \sum_{i=0}^{n} \alpha^{(n-i)} x[i] \]

Since the system sums all previous values of \( x[n] \), consider system output to the "worst-case" input \( x[n] = M_x \).
\[ y[n] = M_x (\alpha^n + \alpha^{n-1} + \ldots + \alpha + 1) = M_x \sum_{i=0}^{n} \alpha^i \]

For any finite \( n \), \( y[n] < \infty \).

When \( n \to \infty \), if \( \alpha < 1 \), the geometric series converges to

\[ \lim_{n \to \infty} y[n] = M_x \frac{1}{1 - \alpha} \]

So, if \( \alpha < 1 \), the system is BIBO stable.
Are following systems BIBO stable?

- $y(t) = x(t^2)$
- $y(t) = tx(t)$
• $y(t) = x(t^2)$
• \( y(t) = x(t^2) \)

If \( x(t) \) is bounded by \( M_x \), i.e. \( |x(t)| < M_x \), then \( x(t^2) \) is also bounded by \( M_x \). As a result, \( y(t) \) is also bounded by \( M_y = M_x \).

This system is \textit{BIBO stable}.
• \( y(t) = tx(t) \)
• $y(t) = tx(t)$

Assume that $x(t)$ is bounded, it is clear that $t$ is unbounded. Hence, $y(t)$ is also unbounded.

This system is not BIBO stable
Example: Cruise control, from introduction,

\[ y = H(k(x - y)) \]

We’ll see later that this system can become unstable if \( k \) is too large (depending on \( H \))

- Positive error adds bolus of gas
- Delay car velocity change, speed overshoots
- Negative error cuts gas off
- Delay in velocity change, speed undershoots
- Repeat!
System Invertability

• A system is invertable if the input signal can be recovered from the output signal.

• If $F$ is an invertable system, and

$$y = Fx$$

then there is an inverse system $F^{INV}$ such that

$$x = F^{INV} y = F^{INV} Fx$$

so $F^{INV} F = I$, the identity operator.
Example: Multipath echo cancellation

Important problem in communications, radar, radio, cell phones.
Generally there will be multiple echoes.

Multipath can be described by a system $y = Fx$

- If we transmit an impulse, we receive multiple delayed impulses.
- One transmitted message gives multiple overlapping messages

We want to find a system $F^{INV}$ that takes the multipath corrupted signal $y$ and recovers $x$

$$F^{INV}y = F^{INV}(Fx) = (F^{INV}F)x = x$$

Often possible if we allow a delay in the output.
Many systems are described by a linear constant coefficient ordinary differential equation (LCCODE):

\[ a_n y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b_m x^{(m)}(t) + \cdots + b_1 x'(t) + b_0 x(t) \]

with given initial conditions

\[ y^{(n-1)}(0), \ldots , y'(0), y(0) \]

(which fixes \( y(t) \), given \( x(t) \))

* \( n \) is called the order of the system
* \( b_0, \ldots , b_m, a_0, \ldots , a_n \) are the coefficients of the system
This is important because LCCODE systems are \textbf{linear} when initial conditions are all zero.

- Many systems can be described this way
- If we can describe a system this way, we know it is linear

Note that an LCCODE gives an \textit{implicit} description of a system.

- It describes how $x(t)$, $y(t)$, and their derivatives interrelate
- It doesn’t give you an explicit solution for $y(t)$ in terms of $x(t)$

Soon we’ll be able to \textit{explicitly} express $y(t)$ in terms of $x(t)$
Examples

Simple examples

• scaling system \((a_0 = 1, \ b_0 = a)\)

\[ y = ax \]

• integrator \((a_1 = 1, \ b_0 = 1)\)

\[ y' = x \]

• differentiator \((a_0 = 1, \ b_1 = 1)\)

\[ y = x' \]

• integrator with feedback (a few slides back, \(a_1 = 1, \ a_0 = a, \ b_0 = 1\))

\[ y' + ay = x \]
By Kirchoff’s voltage law

\[ x - Li' - Ri - y = 0 \]

Using \( i = Cy' \),

\[ x - LCy'' - RCy' - y = 0 \]

or

\[ LCy'' + RCy' + y = x \]

which is an LCCODE. This is a linear system.
This can represent suspension system, or building during earthquake, . . .
• $x(t)$ is displacement of base; $y(t)$ is displacement of mass

• spring force is $k(x - y)$; damping force is $b(x - y)'$

• Newton’s equation is $my'' = b(x - y)' + k(x - y)$

Rewrite as second-order LCCODE

$$my'' + by' + ky = bx' + kx$$

This is a linear system.
Discrete-Time Systems

- Scaling and delay blocks common

- The system equations are *difference equations*

\[ a_0 y[n] + a_1 y[n - 1] + \ldots = b_0 x[n] + b_1 x[n - 1] + \ldots \]

where \( x[n] \) is the input, and \( y[n] \) is the output.
Discrete-Time System Example

- The input into the delay is

\[ e[n] = x[n] - ay[n] \]

- The output is \( y[n] = e[n - 1] \), so

\[ y[n] = x[n - 1] - ay[n - 1]. \]
Summary

• Many system properties (causality, stability, linearity) have important physical interpretations and simple mathematical descriptions.

• LCCDEs are linear systems, analysis is relatively simple.