Solution to RC circuit on Monday

\[ V = V_0 + RC \frac{dV}{dt} \]

Subtract \( V_e \) from both sides: \( V - V_e = RC \frac{dV}{dt} \)

Isolate variables: \( \frac{dV}{dt} = \frac{V_e}{RC} \)

Integrate both sides: \( \int \frac{dV}{dt} = \int \frac{V_e}{RC} dt \)

Exponentiate: \( e^{\frac{V_e}{RC}} \)

Determine \( e^{V_e/RC} \), using the fact that \( 0 \) when \( 0 \):

\[ V_0 = V_e \left( 1 - \frac{t}{RC} \right) \]

Invert both sides (we can do this since \( - \frac{1}{RC} \) will never be zero):

\[ e^{\frac{V_e}{RC}} = \frac{V_e}{V_0} \]

Multiply both sides by \( V_0 \): \( V_0 e^{\frac{V_e}{RC}} = V_0 - V_e \)

... and solve for \( V_e \): \( V_e = V_0 - V_0 e^{\frac{V_e}{RC}} \)

\[ V_0 = V_e \left( 1 - e^{\frac{V_e}{RC}} \right) \]

Lecture 2–ε. Complex Numbers

- The imaginary unit (denoted \( i \)) is the square root of \( -1 \).
- A complex number has the form \( z = x + iy \), where \( x \) and \( y \) are real numbers
  - \( x \) is the real part of \( z \) (\( x = \text{Re}(z) \))
  - \( iy \) is the imaginary part of \( z \) (\( iy = \text{Im}(z) \))
  - \( z^* = x - iy \) is the complex conjugate of \( z \).
  - The norm (magnitude) of \( z \) is \( |z| = (x^2 + y^2)^{1/2} \)
- Addition: \( z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \)
- Multiplication: \( z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \)
- Reciprocals: \( z^{-1} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \)

Polar diagrams

Exponential form:

\[ z = r e^{i \phi} \]

\[ r = \sqrt{x^2 + y^2} \]

\[ \phi = \arctan \frac{y}{x} \]

Exponential form: \( z = r e^{i \phi} \)

\[ e^{i \phi} = \cos \phi + i \sin \phi \]

\[ r e^{i \phi} = r \cos \phi + ir \sin \phi \]

Calculating with exponentials

Multiplication/Division

\[ z_1 \cdot z_2 = r_1 e^{i \phi_1} r_2 e^{i \phi_2} = r_1 r_2 e^{i(\phi_1 + \phi_2)} \]

\[ \frac{z_1}{z_2} = \frac{r_1 e^{i \phi_1}}{r_2 e^{i \phi_2}} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)} \]

Powers of imaginary numbers

\[ (x + iy)^n = (x e^{i \phi})^n = r^n e^{i n \phi} = r^n (\cos n \phi + i \sin n \phi) \]

Roots of unity (solutions to \( z^n = 1 \))

\[ z_1^n = (x + iy)^n = 1 \]

The angle associated with the \( n \)-th root of unity is therefore \( 2\pi n \)

\[ e^{i \frac{2\pi k}{n}} = \cos(k \cdot 2\pi) + i \sin(k \cdot 2\pi) = 1 \]

For any integer \( k \), \( k \)-times this angle is also an \( n \)-th root of unity.
Lecture 2. Properties of Systems

- Linearity
- Time invariance
- Causality
- Memory.

Recall: RC circuit example

\[ \begin{align*}
\frac{dy}{dt} + \alpha y &= \alpha x, \\
\alpha &= \frac{1}{RC}
\end{align*} \]

Assuming \( y(0) = 0 \), we have the input-output relationship

\[ y(t) = \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma \]

Properties of Input-Output Systems

\[ x \rightarrow \quad y \quad y(t) = T[x(t)] \]

**Linearity.** The system is linear if

\[ T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)] \]

\[ T[k x(t)] = k T[x(t)] \quad \text{for any } k, x_1, x_2. \]

Alternatively, if

\[ T[k_1 x_1(t) + k_2 x_2(t)] = k_1 T[x_1(t)] + k_2 T[x_2(t)] \]

for any \( k_1, k_2, x_1, x_2 \).

Linearity of the RC circuit example.

\[ y(t) = T[x(t)] = \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma. \]

\[ T[k_1 x_1(t) + k_2 x_2(t)] = \int_0^t \alpha e^{-\alpha(t-\sigma)} [k_1 x_1(\sigma) + k_2 x_2(\sigma)] d\sigma \]

\[ = k_1 \int_0^t \alpha e^{-\alpha(t-\sigma)} x_1(\sigma) d\sigma + k_2 \int_0^t \alpha e^{-\alpha(t-\sigma)} x_2(\sigma) d\sigma \]

\[ = k_1 T[x_1(t)] + k_2 T[x_2(t)] \quad \Rightarrow \quad \text{LINEAR.} \]
Time Invariance Property:

If \( y(t) = T[x(t)] \), then \( y(t - \tau) = T[x(t - \tau)] \).

In words, a system is T.I. when: given an input-output pair, if we apply a delayed version of the input, the new output is the delayed version of the original output.

Time invariance of RC circuit

\( y(t) = T[x(t)] = \int_0^t h(t - \sigma)x(\sigma)d\sigma \)

Now, apply the delayed input \( \tilde{x}(t) = x(t - \tau) \)

\( T[\tilde{x}(t)] = \int_0^t h(t - \sigma)\tilde{x}(\sigma)d\sigma = \int_0^{t - \tau} h(t - \sigma)x(\sigma - \tau)d\sigma \)

Let’s prove it using the formula \( y(t) = \int_0^t \alpha e^{-\alpha(t-\sigma)}x(\sigma)d\sigma \)

Assume all signals are zero for \( t < 0 \).

Introduce the notation \( h(t) = \alpha e^{-\alpha t} \)

Introduce the notation \( h(t) = \alpha e^{-\alpha t} \)

\( y(t) = \int_0^t h(t - \sigma)x(\sigma)d\sigma \)

Another example: \( y(t) = t x(t) \)

Linear? Yes, easy to show.

Time invariant? No:

\( T[x(t - \tau)] = t x(t - \tau) \)

\( y(t - \tau) = (t - \tau) x(t - \tau) \)

They are different!

Compare for a particular \( x(t) \) →

Time-varying system
Example: amplitude modulation

\[ y(t) = x(t) \cos(\omega t) \]

Used in AM radio!

Again, this is a linear system.

Is it time invariant?

\[ T[x(t - \tau)] = x(t - \tau) \cos(\omega t) \]

\[ y(t - \tau) = x(t - \tau) \cos(\omega t) \]

Only equal if \( \omega \tau = 2k\pi \)

Therefore, it is a time varying system

Notation: LTI = linear, time invariant
LTV = linear, time varying

Causality and memory

- A system is **causal** if \( y(t) \) depends only on \( x(t), t \leq t_0 \).
  (present output only depends on past and present inputs)
- A system is **memoryless** if \( y(t) \) depends only on \( x(t) \).
  (present output only depends on present input).
- Causal, not memoryless: we say it has memory.

**Examples:**
- Delay system \( y(t) = x(t - \tau), \tau > 0 \)
  is causal, and has memory.
- Backward shift system \( y(t) = x(t + \tau), \tau > 0 \)
  is non-causal: output anticipates the input.

Non-causal systems are not physically realizable

Recap: properties of main examples

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<th>Modulator</th>
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