Lecture 15: Inversion of Laplace Transform

November 21, 2011
Review

Last class we introduced Laplace transform:

- Generalizes Fourier transform
- Allows handling of growing signals, unstable systems
- Simplifies analysis of LCCDEs
  (converts differential equations into algebraic equations)
- Important for systems with feedback, control
We defined a complex frequency $s = \sigma + j \omega$

- Oscillation component $j \omega$
- A decay/growth component $\sigma$

Also defined the corresponding complex exponential $e^{st}$

For which $s$ does $f(t)e^{st} \to 0$ as $t \to \infty$?

Defined bilateral Laplace transform:

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt.$$ 

with the inverse:

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$$
Notice that Fourier transform is just the special case of this:

\[ F(j\omega) = F(s) \big|_{s=j\omega} \]

\[
\begin{align*}
\mathcal{F}(s) &= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds \\
\mathcal{L}(s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-st} dt
\end{align*}
\]

Important: Laplace transform is not unique! \( F(s) \) is usually specified along with *region of convergence* (region in complex plane for which \( F(s) \) does not blow up)
We are primarily interested in causal signals (for which \( f(t) = f(t)u(t) \)), so we defined a unilateral Laplace transform:

\[
F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt
\]

The lower limit \( 0^- \) indicates that we include impulses at the origin.

A bilateral Laplace transform can correspond to different signals (causal, anti-causal, or infinite extent) depending on the region of convergence. The \( e^{\sigma t} \) factor that makes the integral converge for a causal signal can make the integral for an anti-causal signal blow up.

If we restrict ourselves to the unilateral transform the Laplace transform is (almost) unique, and we can ignore the region of convergence.
Example.
Consider the Laplace transform of \( f(t) = e^{-at}u(t) \):

\[
F(s) = \int_0^\infty e^{-at} e^{-st} \, dt = \int_0^\infty e^{-(a+s)t} \, dt = \frac{1}{s + a}
\]

provided we can say \( e^{-(s+a)t} \to 0 \) as \( t \to \infty \).

If \( \Re(s + a) = \sigma + a > 0 \):

\[
\left| e^{-(s+a)t} \right| = \left| e^{-(\sigma+j\omega+a)t} \right| = \left| e^{-j\omega t} \right| \left| e^{-(\sigma+a)t} \right| = e^{-(\sigma+a)t}
\]

The region of convergence is then \( \sigma > -a \), or \( \Re s > -a \).

The Laplace transform pair is

\[
e^{-at} \Leftrightarrow \frac{1}{s + a}
\]
This is very similar to the Fourier transform relationship:

\[ e^{-at} \Leftrightarrow \frac{1}{j\omega + a} \]
Example:

- What is the Laplace transform of *unit step* signal? (this expression will be different from Fourier transform of \( u(t) \) !)

- What is the Laplace transform of \( \delta(t) \) ?

- What is the Laplace transform of \( \cos(\omega_0 t) \) ?
Solution:

\[
\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}\,dt = \int_{0}^{\infty} u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-st}\,dt
\]

\[
= \frac{1}{s} \left[ e^{-st} \right]_0^\infty = \frac{1}{s}
\]

\[
\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t)e^{-st}\,dt = e^{-s0} = 1
\]

\[
\mathcal{L}[\cos(\omega_0 t)] = \int_0^\infty \cos(\omega_0 t)e^{-st}\,dt = \frac{1}{2} \int_0^\infty (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-st}\,dt
\]

\[
= \frac{1}{2} \left( \frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right)
\]

\[
= \frac{s}{s^2 + \omega_0^2}
\]
We’ve studied many properties of Fourier series and Fourier Transform (shift, time-stretch, reversal, etc). Same relationships also hold for Laplace Transform.

In particular, the convolution property holds: if \( x(t) \) is an input to the system with impulse response \( h(t) \), then the output of the system

\[
y(t) = h(t) \ast x(t)
\]

can also be expressed as

\[
Y(s) = H(s)X(s)
\]

where \( H(s), X(s), \) and \( Y(s) \) are corresponding Laplace transforms.
**Derivative Theorem:** If signal $f(t)$ is continuous at $t = 0$, then

$$
\mathcal{L}[f'(t)] = sF(s) - f(0).
$$

Except for the (very important) initial condition $f(0)$ this is the same as for the Fourier transform. Note that for the Fourier transform the initial condition showed up in the integral theorem instead.

**Higher-order derivatives:** applying derivative formula twice yields

\[
\begin{align*}
\mathcal{L}[f''(t)] &= s\mathcal{L}(f') - f'(0) \\
&= s(sF(s) - f(0)) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{align*}
\]

Similar formulas hold for $\mathcal{L}[f^{(k)}(t)]$.
These are summarized in the following table:

<table>
<thead>
<tr>
<th>Integration</th>
<th>Differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(t)$</td>
<td>$G(s)$</td>
</tr>
</tbody>
</table>
| \[
\begin{align*}
\int_0^t f(\tau) d\tau & \quad \frac{1}{s} F(s) \\
 f(t) & \quad F(s) \\
 f'(t) & \quad sF(s) - f(0) \\
 f''(t) & \quad s^2 F(s) - sf(0) - f'(0)
\end{align*}
\] | \[
\begin{align*}
\int_0^t f(\tau) d\tau & \quad \frac{1}{s} F(s) \\
 f(t) & \quad F(s) \\
 f'(t) & \quad sF(s) - f(0) \\
 f''(t) & \quad s^2 F(s) - sf(0) - f'(0)
\end{align*}
\] |

Initial Conditions

Divide by $s$

Multiply by $s$, Initial Conditions
Example: \( f(t) = e^t \), so \( f'(t) = e^t \). Then

\[
\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)
\]
\[
= s\mathcal{L}[e^t] - e^0
\]
\[
= s \left( \frac{1}{s - 1} \right) - 1
\]
\[
= \frac{s - (s - 1)}{s - 1}
\]
\[
= \frac{1}{s - 1}
\]

so

\[
\mathcal{L}[f(t)] = \mathcal{L}[f'(t)] = \frac{1}{s - 1}
\]

as we would expect.
Example: Let $F(s)$ be the Laplace transform of $f(t)$. Show that

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$
Solution: We have, by definition

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t) dt$$

Integration by parts:

$$\mathcal{L}[f'(t)] = f(t)e^{-st}\bigg|_0^\infty + s \int_0^\infty e^{-st} f(t) dt$$

$$= \lim_{t \to \infty} f(t)e^{-st} - f(0) + sF(s)$$

If the Laplace transform $F(s)$ of the function $f(t)$ exists, then

$$\lim_{t \to \infty} f(t)e^{-st} = 0$$

Therefore,

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$
Example: If $F(s)$ is the Laplace Transform of $f(t)$, show that

$$
\mathcal{L} \left[ \int_0^t f(\tau) d\tau \right] = \frac{F(s)}{s}
$$
Solution: We set

\[ g(t) = \int_{0}^{t} f(\tau) d\tau, \quad t \geq 0 \]

Then, we have \( g(0) = 0 \) and \( \frac{dg}{dt} = f(t) \). Therefore,

\[
F(s) = \mathcal{L}[f(t)] = \mathcal{L} \left[ \frac{dg}{dt} \right] \\
= s\mathcal{L}[g(t)] - g(0) = s\mathcal{L}[g(t)] \\
= s\mathcal{L} \left[ \int_{0}^{t} f(\tau) d\tau \right]
\]
Inversion of the Laplace transform

Until now we haven’t considered inverting Laplace transform. We have the equation:

\[ f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} \, ds \]

where \( \sigma \) is large enough that \( F(s) \) is defined for \( \Re{s} \geq c \)

- This involves a contour integral in the complex plane.
- Simpler approach: rewrite a rational Laplace transform into simple terms we recognize, and can invert by inspection (partial fraction expansion).
- Computationally the same operations as contour integral.
We are often faced with a specific form of $F(s)$:

$$
F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}
$$

where $b(s)$ and $a(s)$ are polynomials in $s$.

Some terminology:

- Solutions to $b(s) = 0$ are called **zeros** of $F(s)$ (they set $F(s) = 0$)
- Solutions to $a(s) = 0$ are called **poles** of $F(s)$ (they set $F(s) \to \infty$)

The same terminology exists for Fourier transforms that are expressed as a ratio of polynomials in $j\omega$, but is more often used in discussing Laplace transforms.
Partial fraction expansion

\[ F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} \]

Let’s assume (for now)

- no poles are repeated, i.e., all roots of \( a \) have multiplicity one
- \( m < n \)

then we can write \( F \) in the form

\[ F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n} \]

called **partial fraction expansion** of \( F \)

- \( \lambda_1, \ldots, \lambda_n \) are the poles of \( F \)
- the numbers \( r_1, \ldots, r_n \) are called the **residues**
• when $\lambda_k = \lambda_l^*$, $r_k = r_l^*$

**Example:**

$$\frac{s^2 - 2}{s^3 + 3s^2 + 2s} = \frac{-1}{s} + \frac{1}{s + 1} + \frac{1}{s + 2}$$

Let’s check:

$$\frac{-1}{s} + \frac{1}{s + 1} + \frac{1}{s + 2} = \frac{-1(s + 1)(s + 2) + s(s + 2) + s(s + 1)}{s(s + 1)(s + 2)}$$

In partial fraction form, **inverse Laplace transform** is easy:

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1} \left[ \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n} \right]$$

$$= r_1 \mathcal{L}^{-1} \left[ \frac{1}{s - \lambda_1} \right] + \cdots + r_n \mathcal{L}^{-1} \left[ \frac{1}{s - \lambda_n} \right]$$

$$= r_1 e^{\lambda_1 t} + \cdots + r_n e^{\lambda_n t}$$

This is real since whenever the poles are conjugates, the corresponding residues are also.
Finding the partial fraction expansion

Two steps:

- find poles $\lambda_1, \ldots, \lambda_n$ (i.e., factor $a(s)$)
- find residues $r_1, \ldots, r_n$ (several methods)

**Method 1: Clear Fractions, Solve linear equations**

We’ll illustrate for $m = 2$, $n = 3$

\[
\frac{b_0 + b_1 s + b_2 s^2}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)} = \frac{r_1}{s - \lambda_1} + \frac{r_2}{s - \lambda_2} + \frac{r_3}{s - \lambda_3}
\]

Clear denominators:

\[
b_0 + b_1 s + b_2 s^2 = r_1 (s - \lambda_2)(s - \lambda_3) + r_2 (s - \lambda_1)(s - \lambda_3) + r_3 (s - \lambda_1)(s - \lambda_2)
\]
Equate coefficients:

• coefficient of $s^0$:

\[ b_0 = (\lambda_2 \lambda_3) r_1 + (\lambda_1 \lambda_3) r_2 + (\lambda_1 \lambda_2) r_3 \]

• coefficient of $s^1$:

\[ b_1 = (-\lambda_2 - \lambda_3) r_1 + (-\lambda_1 - \lambda_3) r_2 + (-\lambda_1 - \lambda_2) r_3 \]

• coefficient of $s^2$:

\[ b_2 = r_1 + r_2 + r_3 \]

Now solve for $r_1, r_2, r_3$ (three equations in three variables)
Method 2: Heaviside “Cover-up” Procedure

To get $r_1$, multiply both sides by $s - \lambda_1$ to get

\[
\frac{(s - \lambda_1)(b_0 + b_1 s + b_2 s^2)}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)} = r_1 + \frac{r_2(s - \lambda_1)}{s - \lambda_2} + \frac{r_3(s - \lambda_1)}{s - \lambda_3}
\]

cancel $s - \lambda_1$ term on left and set $s = \lambda_1$:

\[
\frac{b_0 + b_1 \lambda_1 + b_2 \lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} = r_1
\]

an explicit formula for $r_1$!

We can get $r_2, r_3$ the same way.
In the general case we have the formula

\[ r_k = (s - \lambda_k)F(s) \bigg|_{s=\lambda_k} \]

which means:

- multiply \( F \) by \( s - \lambda_k \)
- then cancel \( s - \lambda_k \) from numerator and denominator
- then evaluate at \( s = \lambda_k \) to get \( r_k \)
Example:

\[
\frac{s^2 - 2}{s(s + 1)(s + 2)} = \frac{r_1}{s} + \frac{r_2}{s + 1} + \frac{r_3}{s + 2}
\]

- residue \( r_1 \):

\[
r_1 = \left. \left( r_1 + \frac{r_2 s}{s + 1} + \frac{r_3 s}{s + 2} \right) \right|_{s=0} = \frac{s^2 - 2}{(s + 1)(s + 2)} \bigg|_{s=0} = -1
\]

- residue \( r_2 \):

\[
r_2 = \left. \left( \frac{r_1 (s + 1)}{s} + r_2 + \frac{r_3 (s + 1)}{s + 2} \right) \right|_{s=-1} = \frac{s^2 - 2}{s(s + 2)} \bigg|_{s=-1} = 1
\]

- residue \( r_3 \):

\[
r_3 = \left. \left( \frac{r_1 (s + 2)}{s} + \frac{r_2 (s + 2)}{s + 1} + r_3 \right) \right|_{s=-2} = \frac{s^2 - 2}{s(s + 1)} \bigg|_{s=-2} = 1
\]

so we have:

\[
\frac{s^2 - 2}{s(s + 1)(s + 2)} = \frac{-1}{s} + \frac{1}{s + 1} + \frac{1}{s + 2}
\]
Example.
Find the inverse Laplace transforms of the following signals:
Don’t forget to specify ROC!

- \( F(s) = \frac{1}{(s-1)(s+1)} \)
- \( F(s) = \frac{s}{(s+5)(s+4)} \)
Solution:

\[
\frac{1}{(s - 1)(s + 1)} = \frac{1/2}{(s - 1)} + \frac{-1/2}{(s + 1)}
\]

Therefore,

\[
f(t) = \left(\frac{1}{2}e^t - \frac{1}{2}e^{-t}\right)u(t) = \sinh(t)u(t), \quad ROC: \Re\{s\} > 1
\]

\[
\frac{s}{(s + 5)(s + 4)} = \frac{5}{(s + 5)} + \frac{-4}{(s + 4)}
\]

Therefore,

\[
f(t) = (5e^{-5t} - 4e^{-4t})u(t), \quad ROC: \Re\{s\} > -4
\]
The Laplace transforms provide an easy solution for linear differential equations with

- Constant coefficients
- Initial conditions
- Input signals

Solution procedure:

- Laplace transform converts differential equation, initial conditions, and input signals into an algebraic equation
- Solve for the Laplace transform of the output
- Inverse Laplace transform provides the solution
Example

Solve the LCCODE

\[ y''(t) + 5y'(t) + 6y(t) = f'(t) + f(t) \]

where \( y(t) \) is the output signal, and initial conditions are \( y(0^-) = 2 \), and \( y'(0) = 1 \). Assume that the input signal is

\[ f(t) = e^{-4t} u(t). \]

Solution: Taking the Laplace transform of the left side of the equation is

\[
\begin{align*}
  s^2 Y(s) - sy(0^-) - y'(0^-) + 5(sY(s) - y(0^-)) + 6Y(s) \\
  = Y(s)(s^2 + 5s + 6) - 2s - 1 - 10 \\
  = Y(s)(s^2 + 5s + 6) - (2s + 11)
\end{align*}
\]
The Laplace transform of the right side is

\[ sF(s) - f(0^-) + F(s) = F(s)(s + 1) = \frac{s + 1}{s + 4} \]

where we have used the fact that that \( \mathcal{L}[f(t)] = \mathcal{L}[e^{-4t}u(t)] = \frac{1}{s+4} \). The Laplace transform of the equation is then

\[ Y(s)(s^2 + 5s + 6) - (2s + 11) = \frac{s + 1}{s + 4}. \]

Solving for \( Y(s) \),

\[ Y(s) = \frac{2s + 11}{s^2 + 5s + 6} + \frac{s + 1}{(s + 4)(s^2 + 5s + 6)}. \]

from init. cond. from input

The first part can be traced back to the initial conditions, and the second part is due to the input. We'll return to this in a few pages.
Combining these terms we get

\[ Y(s) = \frac{(2s + 11)(s + 4) + (s + 1)}{(s + 4)(s^2 + 5s + 6)} = \frac{2s^2 + 20s + 45}{(s + 4)(s + 3)(s + 2)} \]

To find the inverse Laplace transform we first find the partial expansion,

\[ \frac{2s^2 + 20s + 45}{(s + 2)(s + 3)(s + 4)} = \frac{r_1}{s + 2} + \frac{r_2}{s + 3} + \frac{r_3}{s + 4}. \]

Using the cover up algorithm,

\[ r_1 = \frac{2(-2)^2 + 20(-2) + 45}{(-2 + 4)(-2 + 3)} = \frac{8 - 40 + 45}{(2)(1)} = \frac{13}{2} \]
\[ r_2 = \frac{2(-3)^2 + 20(-3) + 45}{(-3 + 4)(-3 + 2)} = \frac{18 - 60 + 45}{(1)(-1)} = -3 \]
\[ r_3 = \frac{2(-4)^2 + 20(-4) + 45}{(-4 + 3)(-4 + 2)} = \frac{32 - 80 + 45}{(-1)(-2)} = \frac{-3}{2} \]
Then

\[ Y(s) = \frac{13/2}{s + 2} - \frac{3}{s + 3} - \frac{3/2}{s + 4}. \]

The solution \( y(t) \) is then found by taking the term-by-term inverse Laplace transform

\[ y(t) = \left[ \frac{13}{2} e^{-2t} - 3 e^{-3t} - \frac{3}{2} e^{-4t} \right] u(t) \]

Zero-State and Zero-Input Solutions: This solution combines the effect of the initial conditions, and the input. We can also keep them separate, and solve for the zero-input and zero-state signals,

\[
Y(s) = \frac{2s + 11}{s^2 + 5s + 6} + \frac{s + 1}{(s + 4)(s^2 + 5s + 6)}
\]

\[ = \underbrace{\frac{2s + 11}{(s + 2)(s + 3)}}_{\text{zero input}} + \underbrace{\frac{s + 1}{(s + 2)(s + 3)(s + 4)}}_{\text{zero state}}. \]
The zero input component is

\[
\frac{2s + 11}{(s + 2)(s + 3)} = \frac{r_1}{s + 2} + \frac{r_2}{s + 3} = \frac{7}{s + 2} - \frac{5}{s + 3}.
\]

which corresponds to the signal

\[
y_{zi}(t) = \left[7e^{-2t} - 5e^{-3t}\right] u(t)
\]

The zero state component is

\[
\frac{s + 1}{(s + 2)(s + 3)(s + 4)} = \frac{r_1}{s + 2} + \frac{r_2}{s + 3} + \frac{r_3}{s + 4} = -\frac{1/2}{s + 2} + \frac{2}{s + 3} - \frac{3/2}{s + 4}
\]
which corresponds to the signal

\[ y_{zs}(t) = \left[ -\frac{3}{2}e^{-4t} + 2e^{-3t} - \frac{1}{2}e^{-2t} \right] u(t) \]

The complete solution is then

\[
y(t) = y_{zi}(t) + y_{zs}(t) \\
= \left[ 7e^{-2t} - 5e^{-3t} \right] u(t) + \left[ -\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t} \right] u(t) \\
= \left[ \frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t} \right] u(t)
\]
Zero-State Response: Transfer Function of LTI Systems

A linear time invariant system is complete characterized by its impulse response. If the impulse response is \( h(t) \), the output \( y(t) \) for an input \( x(t) \) is

\[
y(t) = (h \ast x)(t).
\]

By the convolution theorem, the Laplace transform of this equation is

\[
Y(s) = H(s)X(s)
\]

The function \( H(s) \) is the transfer function of the system.

For a system described by a LCCODE, the system equation is of the form

\[
a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_1 x(t) + b_0 x(t)
\]
If the initial conditions are all zero (zero-state solution), the Laplace transform of the equation is

\[ Y(s) \left[ a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 \right] = X(s) \left[ b_m s^m + b_{m-1} s^{m-1} + \cdots + a_0 \right] \]

Solving for \( Y(s) \)

\[ Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} X(s) \]

The transfer function is then

\[ H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \]

and the impulse response of the system is found by the inverse Laplace transform.