Systems and Signals

Lecture 14: Discrete Time Fourier Transform

November 17, 2010
Administration

- MATLAB Project due last day of class.
- Homework 6 due Nov.24 (has been posted Monday)
- Reschedule another lecture.
Frequency representation review

So far we analyzed frequency spectrum of

• Continuous-time periodic signals (Continuous Time Fourier Series - CTFS)
  – spectrum consists of multiples of fundamental frequency $\omega_0$
  – potentially infinite number of frequencies present
  – potentially aperiodic frequency representation

• Discrete time periodic signals (Discrete Time Fourier Series - DTFS)
  – spectrum consists of multiples of $\omega_0$
  – spectrum always periodic
    if time domain is periodic in $N$, then so is frequency domain representation
– only $N$ frequency coefficients are needed to completely specify time-domain signal

- Continuous-time aperiodic signals (Continuous Time Fourier Transform - CTFT)
  – spectrum is continuous (function of $\omega$)
  – infinite number of frequencies present

It remains to discuss discrete-time aperiodic signals

Use *Discrete Time Fourier Transform (DTFT)* to analyze these signals

- spectrum is continuous (function of $e^{j\omega}$)
- periodic in $2\pi$

In general very similar to CTFT - many properties identical
A diagram to remember

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Define a signal $x[n]$, with $x[n] = 0$ for $n > N_2$, $n < -N_1$.
Create also a periodic extension $\tilde{x}[n]$ with period $N$ (shown below).

As period $N \to \infty$, $\tilde{x}[n] = x[n]$ for any finite $n$.

As we know, $\tilde{x}[n]$ has a DTFS representation:

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{j k \frac{2\pi}{N} n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-j k \frac{2\pi}{N} n}$$
on the interval $-N_1 \leq n \leq N_2$, $\tilde{x}[n] = x[n]$, and $x[n]$ is zero elsewhere, so we can write

$$a_k = \frac{1}{N} \sum_{n=-N}^{N} x[n] e^{-jk \frac{2\pi}{N} n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j \frac{2\pi}{N} n n}$$

(since $x[n]$ is zero outside $[-N_1, N_2]$)

Define a function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \omega n}$$

We see that

$$a_k = \frac{1}{N} X(e^{j \frac{2\pi}{N} k})$$

Coefficients $a_k$ are proportional to samples of the newly defined function.
Furthermore, we can recover our periodic signal $\tilde{x}[n]$ from these samples

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} X(e^{j\frac{2\pi}{N}k})e^{j\frac{2\pi}{N}kn}$$

or since $\frac{1}{N} = \frac{\omega_0}{2\pi}$

$$\tilde{x}[n] = \frac{\omega_0}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0})e^{jk\omega_0n}$$

As $N$ increases, $\omega_0$ decreases. As $\omega_0 \to 0$, ($N \to \infty$), this sum becomes an integral. We can write the expression for the aperiodic signal:

$$x[n] = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X(e^{j\omega})e^{j\omega n} d\omega$$

where $X(e^{j\omega})$ is the frequency spectrum of a discrete time signal
Discrete Time Fourier Transform

We now have a relationship between frequency spectrum of discrete-time aperiodic signal and its time-domain representation:

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \text{ (Analysis)} \]

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \text{ (Synthesis)} \]

Notice that frequency spectrum \( X(e^{j\omega}) \) is periodic in \( \omega = 2\pi \) and so is \( e^{j\omega n} \), so \( X(e^{j\omega})e^{j\omega n} \) is also periodic.

Notice also - the integral in synthesis equation is over \( 2\pi \).

Remember we said before that a signal periodic in \( N \) can be expressed using DTFS as a linear combination of \( N \) complex exponentials.
An aperiodic signal can be viewed as a linear combination of infinite number of complex exponentials, whose frequencies are infinitesimally close.

Equations for DTFT and CTFT are similar; we expect many properties of CTFT to hold true for DTFT as well. In fact, they do - but there are several key differences.

Main difference: $X(e^{j\omega})$ is periodic in $2\pi$

Discrete-time complex exponentials $e^{j\omega n}$ are periodic in $2\pi$. 
This periodicity has some consequences

- Frequencies close to $\omega = 0$ or $\omega = 2\pi k$, $k = \pm 1, \pm 2, \ldots$ are considered "low". Signals at these frequencies are varying slowly.

- Frequencies close to $\omega = \pi$ or $\omega = \pi k$, $k = \pm 1, \pm 3, \pm 5, \ldots$ are considered "high". Signals at these frequencies are varying quickly.

Consider $x[n] = \cos(\pi n)$ (fastest discrete-time signal) and $x[n] = \cos(\frac{\pi}{16} n)$ as examples.
Notation.

As before, we use the shorthand notation:

\[ X(e^{j\omega}) = \mathcal{F}\{x[n]\} \]

\[ x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\} \]

We refer to Discrete Time Fourier Transform as DTFT (or just FT when discreteness is clear from context)
Example.
What is \( X(e^{j\omega}) \) of \( \delta[n] \)?

\[
X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}
\]

\[
X(e^{j\omega}) = \sum_n \delta[n]e^{-j\omega n}
\]

by the sifting property

\[
X(e^{j\omega}) = e^{-j\omega(0)}
\]

\[
X(e^{j\omega}) = 1
\]

This result is identical to CTFT result \( \delta(t) \leftrightarrow 1 \)
Example.
What is the inverse DTFT of an impulse train of delta functions separated by $2\pi$:

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Use the synthesis equation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Integrate from $0^-$ to $2\pi^-$. On this interval only the delta at the origin is present in $X(e^{j\omega})$.

$$x[n] = \frac{1}{2\pi} \int_{0^-}^{2\pi^-} 2\pi \delta(\omega) e^{j\omega n} d\omega$$

$$x[n] = \frac{2\pi}{2\pi} e^{j(0)n} = 1$$
This result is also similar to the continuous-time relationship $2\pi\delta(\omega) \leftrightarrow 1$ that we’ve seen before.

**Example.**
What is the Fourier transform of $x[n] = a^n u[n]$, $|a| < 1$?

Use the analysis equation again:

$$X(e^{j\omega}) = \sum_n a^n u[n] e^{-j\omega n}$$

Modify limits and recognize $a^n e^{-j\omega n} = (ae^{-j\omega})^n$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$
Use the geometric series formula \( \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \) and write:

\[
X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}
\]

What is \( X(e^{j(0)}) \)?

What is \( X(e^{j(\pi)}) \)?

Is this a low-pass or a high-pass filter?
Example.
What is $x[n] = a^n$, if $|a| < 1$?

Rewrite $x[n] = a^n u[n] + a^{-n} u[-n - 1]$

$$X(e^{j\omega}) = \sum_n \left( a^n u[n] + a^{-n} u[-n - 1] \right) e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n}$$

The first term was evaluated in the previous example. Make the substitution $m = -n$ in the second term:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m$$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$
This can be further simplified to

\[ X(e^{j\omega}) = \frac{1 - a^2}{1 - 2a \cos(\omega) + a^2} \]

Notice that \( x[n] \) and \( X(e^{j\omega}) \) are both real and even signals. This would be expected of continuous time Fourier transform pairs - same property holds in DTFT case.
Example.
What is $X(e^{j\omega})$ if time domain signal $x[n]$ is:

$$x[n] = \begin{cases} 
1 & |n| \leq N_1 \\
0 & \text{otherwise}
\end{cases}.$$

In deriving the result, you might find helpful the geometric series formula:

$$\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1 - r}$$
DTFT of Periodic Signals

In continuous time, we have a transform pair \( e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0) \)

We expect a similar result to hold in discrete-time.

Indeed, if

\[
X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l)
\]

\[
x[n] = \frac{1}{2\pi} \int_{2\pi}^{0} \left[ \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l) \right] e^{j\omega n} d\omega
\]

As before, choose to integrate from \( 0^- \) to \( 2\pi^- \), so the infinite sum over \( l \) can be reduced to \( \delta(\omega - \omega_0) \).

\[
x[n] = \int_{0^-}^{2\pi^-} \delta(\omega - \omega_0)e^{j\omega n} d\omega = e^{j\omega_0 n}
\]
Notice that instead of a single delta in frequency domain (as in continuous-time), we now had an infinite train of delta functions separated by $2\pi$.

Consider now an arbitrary periodic signal $x[n]$ with DTFS representation:

$$x[n] = \sum_{k=-N}^{N} a_k e^{j k \frac{2\pi}{N} n}$$

We will show that frequency spectrum $X(e^{j\omega})$ consists of delta functions separated by $\frac{2\pi}{N}$ and scaled by $2\pi a_k$ (and this spectrum is periodic)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$
Consider the synthesis equation

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \]

Substitute the scaled train of delta functions for \( X(e^{j\omega}) \)

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})e^{j\omega n} d\omega \]

After specifying limits of integration, we also specify limits on the infinite sum

\[ x[n] = \int_{-\pi}^{\pi} \sum_{k=0}^{N-1} a_k \delta(\omega - \frac{2\pi k}{N})e^{j\omega n} d\omega \]

\[ x[n] = \sum_{k=0}^{N-1} a_k \int_{0}^{\frac{2\pi}{N}} \delta(\omega - \frac{2\pi k}{N})e^{j\omega n} d\omega \]
\[ x[n] = \sum_{k=0}^{N-1} a_k e^{j k \frac{2\pi}{N} n} \]

which is what was required to be demonstrated.

Note also: we could have chosen different limits of integration (not 0-2\(\pi\)) and different range of the sum and arrived at the same result.

Thus, if \( x[n] \) is periodic and has Fourier series coefficients \( a_k \), its Discrete Time Fourier Transform is:

\[
X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})
\]
Example.
If \( x[n] = \cos(\omega_0 n) \), what is \( X(e^{j\omega}) \)?

\[
x[n] = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}
\]

From our properties,

\[
X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi \delta(\omega - \omega_0 + 2\pi l) + \sum_{l=-\infty}^{\infty} \pi \delta(\omega + \omega_0 + 2\pi l)
\]

Infinite sums over \( l \) reflect the \( 2\pi \) periodicity of frequency domain.
Properties of Discrete-Time Fourier Transform

Several properties are identical to CTFT; we’ve seen them before:

**Linearity**: \( A x_1[n] + B x_2[n] \leftrightarrow AX_1(e^{j\omega}) + BX_2(e^{j\omega}) \)

**Time Shift**: \( x[n - n_0] \leftrightarrow X(e^{j\omega})e^{-j\omega n_0} \)

**Frequency Shift**: \( x[n]e^{j\omega_0 n} \leftrightarrow X(e^{j(\omega - \omega_0)}) \)

**Conjugate symmetry**: if \( x[n] = x^*[n] \), \( X(e^{j\omega}) = X^*(e^{-j\omega}) \)

From this it also follows:

- \( |X(e^{j\omega})| = |X(e^{-j\omega})| \) (magnitude is even)
- \( <X(e^{j\omega}) = - <X(e^{-j\omega}) \) (phase is odd)

**Time Reversal**: \( x[-n] \leftrightarrow X(e^{-j\omega}) \)

**Frequency differentiation**: \( nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega} \)

**Parseval relationship**: \( \sum_{m=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \)
**Time Differencing:**

\[ x[n] - x[n-1] \leftrightarrow (1 - e^{-j\omega})X(e^{j\omega}) \]

This is easy to derive using time-shift property.

Analog of time-derivative property of CTFT - very useful

**Accumulation:**

\[
\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j(0)}) \sum_{k} \delta(\omega - 2\pi k)
\]

This is similar to the integration property in CTFT. As before, the infinite sum over \(k\) reflects periodicity of the frequency domain.
Time Expansion
Recall that in continuous-time we had a time-scaling property:

\[ x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \]

In discrete time, it's problematic to define signals like \( x[an] \) where \( a \) is not an integer.

for example, what is \( x[1.5n] \) ???

We define a specific form of time-expanded signals:

\[ x(k)[n] = \begin{cases} 
  x[n/k] & \text{if } n \text{ is a multiple of } k \\
  0 & \text{otherwise}
\end{cases} \]

In the description above, \( k \) is an integer. For clarity of description, we show a pair of signals: \( x[n] \) and \( x(2)[n] \) below:
Intuitively, $x(2)[n]$ is slowed down relative to $x[n]$.

It is desirable to express $x(k)[n] \leftrightarrow X(k)(e^{j\omega})$ in terms of $X(e^{j\omega})$:

$$X(k)(e^{j\omega}) = \sum_{n} x(k)[n] e^{-j\omega n}$$

since $x(k)[n] = 0$ for all $n$ that aren’t multiples of $k$, we can write:

$$X(k)(e^{j\omega}) = \sum_{n} x(k)[nk] e^{-j\omega nk}$$
Further, since \( x(k)[nk] = x[n] \)

\[
X(k)(e^{j\omega}) = \sum_n x[n]e^{-j\omega nk}
\]

Thus,

\[
X(k)(e^{j\omega}) = X(e^{jk\omega})
\]

\[
x(k)[n] \leftrightarrow X(e^{jk\omega})
\]

Conceptually, the relationship is similar to CTFT case: when a signal is slowed-down in time-domain, frequency domain is compressed.
Convolution

\[ y[n] = x[n] * h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \]

Multiplication

\[ y[n] = x[n]h[n] \leftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta \]

This is very similar to result we’ve seen before - but not quite.

Convolution is only over \(2\pi\) (single period), not over entire real line. This is a periodic convolution (since \(X(e^{j\omega}), H(e^{j\omega})\) are periodic).
Example.
What is $H(e^{j\omega})$ of two-point moving average system:

$$h[n] = \delta[n] - \delta[n - 1]$$

this is a simple discrete-time low-pass filter

Example.
If $x[n] \leftrightarrow X(e^{j\omega})$, express $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$:

a) $y[n] = (n - 1)^2 x[n]$

b) $y[n] = \frac{1}{2} x^*[-n] + \frac{1}{2} x[n]$
Example.
Given a low-pass filter with transfer function $H(e^{j\omega})$ and impulse response $h[n]$: 

$$H(e^{j\omega}) = \begin{cases} 
1 & \text{if } |\omega| < \frac{\pi}{2} \\
0 & \text{otherwise} 
\end{cases}$$

a) What is the output when input is $x[n] = 1$?
b) What if $x[n] = \cos(\pi n)$?

Consider another system $G$, whose impulse response $g[n]$ is related to response of system $H$ by $g[n] = h[n]e^{j\pi n}$.

What is the output when the inputs are $x[n] = 1$, $x[n] = \cos(\pi n)$?

Hint: What is $G(e^{j\omega})$?
Discrete-time systems characterized by LCCDE

Input-output relationship of many LTI systems can be expressed in form of linear constant coefficient difference equation:

\[
\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]
\]

Taking the Fourier transform of both sides:

\[
\mathfrak{F}\left\{ \sum_{k=0}^{N} a_k y[n - k] \right\} = \mathfrak{F}\left\{ \sum_{k=0}^{M} b_k x[n - k] \right\}
\]

\[
\sum_{k=0}^{N} a_k \mathfrak{F}\left\{ y[n - k] \right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{ x[n - k] \right\}
\]
\[
\sum_{k=0}^{N} a_k Y(e^{j\omega}) e^{-j k \omega} = \sum_{k=0}^{M} b_k X(e^{j\omega}) e^{-j k \omega}
\]

After factoring:

\[
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j k \omega}}{\sum_{k=0}^{N} a_k e^{-j k \omega}}
\]

This is a rational function.
You should think of this as a polynomial in \(e^{j\omega}\) (rather than a polynomial in \(j\omega\) as in continuous-time case).
Example.
Suppose you are given a difference equation description of a causal LTI system:

\[ y[n] - ay[n - 1] = x[n] \]

What is \( h[n] ? \)

We use DTFT to write:

\[ Y(e^{j\omega}) - aY(e^{j\omega})e^{-j\omega} = X(e^{j\omega}) \]

\[ (1 - ae^{-j\omega})Y(e^{j\omega}) = X(e^{j\omega}) \]

\[ H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \]

this is readily recognized as:

\[ h[n] = a^n u[n] \]
Example.
The difference equation below corresponds to a causal LTI system. Find $h[n]$. 

$$y[n] - \frac{3}{4}y[n - 1] + \frac{1}{8}y[n - 2] = 2x[n]$$

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

Using partial fractions:

$$H(e^{j\omega}) = \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\omega}}$$
Solving \( A(1 - \frac{1}{4}e^{-j\omega}) + B(1 - \frac{1}{2}e^{-j\omega}) = 2 \), \( A = 4 \) and \( B = -2 \)

\[
H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}
\]

This is recognized as:

\[
h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]
\]

Conclusion:
Procedure for solving difference equations is very similar to continuous-time case. First find \( H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \), then use partial fractions to express polynomial in terms of first-order terms, and finally find inverse Fourier transform.