Systems and Signals

Lecture 13: Introduction to Laplace Transform

November 09, 2011
Administration

• Review Session: Monday, Nov. 14.

• Second Midterm: Wednesday, Nov. 16, 10:00am-12:00pm in class.

• No office hours on Tuesday, Nov. 15.

• TA extra office hour on Tuesday, Nov. 15, 4:00pm-6:00pm. Location: TBA.
Agenda

Today’s topics

• Laplace Transform
Limitations of the Fourier Transform

So far we’ve considered Fourier Transforms (and Fourier series) of discrete and continuous signals.

- To be useful, Fourier transforms must exist, or be defined in a generalized sense.
- For many areas, this will be all you will need (communications, optics, image processing).

For many signals and systems the Fourier transform is not enough:

- Signals that grow with time (your bank account, or the GDP of the US)
- Systems that are unstable (many mechanical or electrical systems).

These are important problems. How can we analyze these?
Consider the signal
\[ f(t) = e^{2t}u(t) \]
This is an increasing exponential, so it doesn’t have a Fourier transform.

However, we can create a new function
\[ \phi(t) = f(t)e^{-\sigma t}. \]
If \( \sigma > 2 \), then this is a decreasing exponential. It does have a Fourier transform.

The Fourier transform represents \( \phi(t) \) in terms of spectral components \( e^{j\omega t} \).

We can express \( f(t) \) as
\[ f(t) = \phi(t)e^{\sigma t} \]
Each spectral component is multiplied by \( e^{\sigma t} \), so \( f(t) \) can be represented by spectral components \( e^{\sigma t}e^{j\omega t} = e^{(\sigma+j\omega)t} \).
\[ \phi(t) = f(t)e^{-\alpha t} \]
How big should we choose $\sigma$?

For $f(t) = e^{2t}$, any $\sigma > 2$ will produce a decaying, Fourier transformable signal, and a different Fourier transform.

If $\sigma_0$ is the smallest value for which $f(t)e^{-\sigma_0 t}$ converges to zero, then any $\sigma > \sigma_0$ will do.

This means the spectrum of $f(t)$ is not unique. The part of the complex plane where the spectrum exists is the *region of convergence*. 
Bilateral Laplace Transform

The Fourier transform is:

\[ F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \]

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \]

The Fourier transform of \( f(t)e^{-\sigma t} \) is

\[ \mathcal{F} [f(t)e^{-\sigma t}] = \int_{-\infty}^{\infty} f(t)e^{-\sigma t} e^{-j\omega t} dt \]

\[ = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t} dt = \int_{-\infty}^{\infty} f(t)e^{-st} dt \]

\[ = F(s) \]
were \( s \) the complex frequency \( s = \sigma + j\omega \).

The complex frequency \( s \) includes:

- the oscillation component \( j\omega \) that we’re used to, plus
- a decay/growth component \( \sigma \).

This is not what you intuitively think of as a frequency.
This is the *bilateral Laplace transform*, which is defined as

\[ F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt. \]

The inverse can be shown to be

\[ f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \]

where \( c > \sigma_0 \) (but we will never use this!).

The notation for the Laplace transform is

\[ F(s) = \mathcal{L}[f(t)] \quad \text{and} \quad f(t) = \mathcal{L}^{-1}[F(s)] \]

or simply

\[ f(t) \Leftrightarrow F(s). \]
The Fourier transform is a special case of the Laplace transform

\[ F(j\omega) = F(s)\big|_{s=j\omega} \]

provided the \( j\omega \) axis is in the region of convergence.

The Fourier transform is the Laplace transform evaluated along the \( j\omega \) axis. It is also often called the frequency response.

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t} d\omega \]

\[ f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds \]
The main motivation for using the Laplace transform is that it converts \textit{integral} and \textit{differential} equations into \textit{algebraic} equations.

Similar to the Fourier transform (which is a special case), but

\begin{itemize}
  \item Handles growing signals and unstable systems,
  \item Easily includes non-steady-state conditions.
\end{itemize}

Allows us to analyze

\begin{itemize}
  \item LCCODEs
  \item Complicated circuits with sources, Ls, Rs, and Cs
  \item Complicated systems with integrators, differentiators, gains.
\end{itemize}
The Unilateral Laplace Transform

We’ll be interested in causal signals. This is the most common application for Laplace transforms.

The bilateral Laplace transform of a causal signal \( f(t)u(t) \) is

\[
F(s) = \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt
\]

\[
= \int_{0^-}^{\infty} f(t)e^{-st}dt.
\]

This is the *unilateral Laplace transform*.

The lower limit \( 0^- \) indicates that we include impulses at the origin (we could exclude them, and make the limit \( 0^+ \), but this would create another less useful transform).
The unilateral Laplace transform is just the bilateral transform for causal signals.

The same bilateral Laplace transform can correspond to different signals (causal, anti-causal, or infinite extent) depending on the region of convergence. The $e^{\sigma t}$ factor that makes the integral converge for a causal signal can make the integral for an anti-causal signal blow up.

If we restrict ourselves to the unilateral transform the Laplace transform is (almost) unique, and we can ignore the region of convergence.
Example: Similar Fourier and Laplace Transforms

Since the Fourier and Laplace transforms are so closely related, we might expect that the Fourier and Laplace transforms of particular functions would be similar. This is sometimes, but not always, true.

Consider the Laplace transform of \( f(t) = e^{-at}u(t) \):

\[
F(s) = \int_0^\infty e^{-at} e^{-st} \, dt = \int_0^\infty e^{-(a+s)t} \, dt = -\frac{1}{s + a} e^{-(s+a)t} \bigg|_0^\infty = \frac{1}{s + a}
\]

provided we can say \( e^{-(s+a)t} \to 0 \) as \( t \to \infty \). If \( \Re(s + a) = \sigma + a > 0 \):

\[
\left| e^{-(s+a)t} \right| = \left| e^{-(\sigma+j\omega+a)t} \right| = \left| e^{-j\omega t} \right| \left| e^{-(\sigma+a)t} \right| = e^{-(\sigma+a)t}
\]

The region of convergence is then \( \sigma > -a \), or \( \Re s > -a \).
The Laplace transform pair is

\[ e^{-at} \iff \frac{1}{s + a} \]

The Fourier transform of \( e^{-at} \) for \( a > 0 \),

\[ \mathcal{F}[e^{-at}] \iff \frac{1}{j\omega + a}. \]

The Laplace transform is the Fourier transform with \( j\omega \) replaced by \( s \). This is true only when the region of convergence includes the \( j\omega \) axis.

The Laplace transform holds for any \( a \), positive or negative. There is always some \( s \) such that \( \Re s > -a \). For example, if \( \lambda > 0 \),

\[ \mathcal{L}[e^{\lambda t}] = \frac{1}{s - \lambda} \]

This doesn’t have a Fourier transform!
Examples: Different Fourier and Laplace Transforms

Other signals have very different Fourier and Laplace transforms. This is true of signals that have generalized Fourier transforms (or no transform at all!). Some of these are:

**unit step:** $u(t) = 1$ for $t \geq 0$, and 0 otherwise. Same as constant for causal signals!

$$F(s) = \int_{0}^{\infty} e^{-st} \, dt = \frac{-1}{s}e^{-st}\bigg|_{0}^{\infty} = \frac{1}{s}$$

provided we can say $e^{-st} \to 0$ as $t \to \infty$. If $\Re s > 0$

$$|e^{-st}| = \left\{ e^{-j(\Im s)t} \right\} |e^{-s}t| = e^{-(\Re s)t}$$

so region of convergence is $\Re s > 0$
We have the transform pair

\[ u(t) \Leftrightarrow \frac{1}{s} \]

In this case the Laplace and Fourier transforms are very different. The Fourier transform of the unit step is

\[ \mathcal{F}[u(t)] = \pi \delta(\omega) + \frac{1}{j\omega} \]

This is almost the same as the Laplace transform with \( s = j\omega \), but has the additional \( \pi \delta(\omega) \) term.

The Laplace transform does not converge for \( \Re s = 0 \) (along the \( j\omega \) axis), and is only defined for \( \Re s > 0 \).

Recall that the Fourier transform of \( u(t) \) existed only in the limit, as a generalized transform.

The Laplace transform deals with \( u(t) \) easily, where we had to work to get a usable Fourier transform.
Cosine: first express \( f(t) = \cos \omega t \) as

\[
f(t) = \frac{1}{2} \left[ e^{j\omega t} + e^{-j\omega t} \right]
\]

Now we can find \( F(s) \) as

\[
F(s) = \int_{0}^{\infty} \frac{1}{2} \left[ e^{j\omega t} + e^{-j\omega t} \right] e^{-st} dt
\]

\[
= \frac{1}{2} \int_{0}^{\infty} e^{(-s+j\omega)t} + e^{(-s-j\omega)t} dt
\]

\[
= \frac{1}{2} \left( \frac{1}{s-j\omega} + \frac{1}{s+j\omega} \right)
\]

\[
= \frac{s}{s^2 + \omega^2}
\]

for \( \Re{s} > 0 \).
The transform pair is

\[ \cos(\omega t) \Leftrightarrow \frac{s}{s^2 + \omega^2} \]

Here again the Fourier transform and the Laplace transform differ. Again, the \( j\omega \) axis is not in the region of convergence for the Laplace transform, and the Fourier transform only exists in the generalized sense.

**Powers of \( t \):** \( f(t) = t^n \) \( (n \geq 1) \)

\[ F(s) = \int_0^\infty t^n e^{-st} \, dt \]
Integrating by parts \( u(t) = t^n \) and \( v'(t) = e^{-st} \)

\[
F(s) = t^n \left( -\frac{e^{-st}}{s} \right) \bigg|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt
\]

\[
= \frac{n}{s} \mathcal{L}(t^{n-1})
\]

provided \( t^n e^{-st} \to 0 \) if \( t \to \infty \), which is true for \( \Re{s} > 0 \)

Applying the formula recursively, we obtain

\[
F(s) = \frac{n!}{s^{n+1}}
\]

which is valid for \( \Re{s} > 0 \).

The transform pair is then

\[
t^n \iff \frac{n!}{s^{n+1}}
\]

This function doesn’t have a Fourier transform!
Impulses at $t = 0$

If $f(t)$ contains impulses at $t = 0$ we choose to include them in the integral defining $F$:

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} \, dt$$

The Laplace transform of $\delta(t)$ is then

$$F(s) = \int_{0-}^{\infty} \delta(t) e^{-st} \, dt = e^{-st} \bigg|_{t=0} = 1$$
We can summarize these in the following table:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}t^2 u(t)$</td>
<td>$\frac{1}{s^3}$</td>
</tr>
<tr>
<td>$t \ u(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\delta'(t)$</td>
<td>$s$</td>
</tr>
<tr>
<td>$\delta''(t)$</td>
<td>$s^2$</td>
</tr>
</tbody>
</table>

**Integration**

- Divide by $s$

**Differentiation**

- Multiply by $s$
Finding the Laplace transform

You should *know* the Laplace transforms of some basic signals, e.g.,

- unit step ($F(s) = 1/s$), impulse function ($F(s) = 1$)
- exponential: $\mathcal{L}[e^{\lambda t}] = 1/(s - \lambda)$
- sinusoids $\mathcal{L}[\cos(\omega t)] = s/(s^2 + \omega^2)$, $\mathcal{L}[\sin(\omega t)] = \omega/(s^2 + \omega^2)$

These, combined with a table of Laplace transforms and the properties (similar to Fourier Transform - we will talk about them later) will get you pretty far.

As always, integrate as a last resort.
Example:
We consider a signal that’s the sum of two exponentials. This signal has a Fourier transform.

\[ f(t) = 3e^{-2t}u(t) - 2e^{-t}u(t) \]

The Laplace transform is:

\[ F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \]

\[ F(s) = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}dt \]

\[ F(s) = 3 \int_{0}^{\infty} e^{-(2+s)t}dt - 2 \int_{0}^{\infty} e^{-(1+s)t}dt \]

\[ F(s) = \frac{3}{s + 2} - \frac{2}{s + 1} \]

The first term converges when \( \Re(s) > -2 \) and the second term converges when \( \Re(s) > -1 \).
Since both terms will converge when $\Re(s) > -1$, ROC of $F(s)$ is $\Re(s) > -1$.

In general you choose the *intersection* of ROCs: regions where all terms converge.
System Function

Example: The input-output transformation of a linear time-invariant and causal system, with input $x(t)u(t)$ can be expressed as

$$y(t) = h(t) \ast x(t) = \int_{0}^{t} h(t - \tau)x(\tau)d\tau, \quad t \geq 0$$

Let $Y(s)$, $H(s)$, and $X(s)$ be the Laplace transforms of $y(t)$, $h(t)$, and $x(t)$ respectively. Prove that

$$Y(s) = H(s)X(s)$$
Proof: By definition, we have

\[
Y(s) = \int_0^{\infty} e^{-st} \left\{ \int_0^t h(t-\tau)x(\tau)d\tau \right\} dt = \int_0^{\infty} e^{-st} \left\{ \int_0^{\infty} h(t-\tau)u(t-\tau)x(\tau)d\tau \right\} dt
\]

Interchanging the order of integration, we obtain

\[
Y(s) = \int_0^{\infty} \left\{ \int_0^{\infty} e^{-st} h(t-\tau)u(t-\tau)dt \right\} x(\tau)d\tau
\]

Setting \( z = t - \tau \), for \( \tau \geq 0 \), the inner integral becomes

\[
e^{-s\tau} \int_{-\tau}^{\infty} e^{-sz} h(z)u(z)dz = e^{-s\tau} \int_0^{\infty} e^{-sz} h(z)dz
\]
As a result,

\[
Y(s) = \int_0^\infty e^{-s\tau} \left\{ \int_0^\infty e^{-sz} h(z) \, dz \right\} x(\tau) \, d\tau
\]

\[
= \int_0^\infty e^{-s\tau} H(s) x(\tau) \, d\tau
\]

\[
= H(s) X(s)
\]

The convolution property holds for Laplace Transform!
The function \( H(s) = \mathcal{L}[h(t)u(t)] \) is called the system function of a linear time-invariant and causal system. Particularly, if the system is single-input single-output (SISO), then

\[
H(s) = \frac{Y(s)}{X(s)},
\]

where \( X(s) \) is the Laplace transform of an input, while \( Y(s) \) is the Laplace transform of the corresponding output.

The system function \( H(s) \) can also be defined as

\[
H(s) = \frac{\text{output of system due to input } e^{st}}{e^{st}}, \quad -\infty < t < \infty
\]
Example: Let $x(t) = e^{st}$ be the input of a linear time-invariant and causal system. Prove that the output of the system is

$$y(t) = e^{st} H(s)$$
Proof: By definition,

\[ y(t) = \int_{-\infty}^{\infty} h(t - \tau) x(\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \]

If the system is causal, then \( h(t) = h(t)u(t) \). We obtain

\[ y(t) = \int_{-\infty}^{\infty} h(\tau)u(\tau)x(t - \tau) d\tau = \int_{0}^{\infty} h(\tau)x(t - \tau) d\tau \]

If \( x(t) = e^{st} \) for all \( t \), then

\[ y(t) = \int_{0}^{\infty} h(\tau)e^{s(t-\tau)} d\tau = e^{st} \int_{0}^{\infty} e^{-s\tau} h(\tau) d\tau \]

Finally,

\[ y(t) = e^{st}H(s) \]
Example: Find the Laplace transform of $t^n f(t)$ in terms of $F(s)$
Solution:

\[ F(s) = \int_0^\infty e^{-st} f(t) dt \]

Therefore,

\[ \frac{dF(s)}{ds} = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt \]

\[ = \int_0^\infty \frac{d}{ds} e^{-st} f(t) dt \]

\[ = \int_0^\infty e^{-st} (-t f(t)) dt = -\mathcal{L}[tf(t)] \]

Then,

\[ \mathcal{L}[tf(t)] = -\frac{dF(s)}{ds} \]
In general, for \( n \geq 0 \):

\[
\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}
\]
Example: Compute

\[ y(t) = \int_{-\infty}^{\infty} \left[ \delta(t - \tau) - e^{-(t-\tau)}u(t - \tau) \right] \tau u(\tau) d\tau \]
Solution:

\[ h(t) = \delta(t) - e^{-t}u(t) \quad \Leftrightarrow \quad H(s) = 1 - \frac{1}{s + 1} = \frac{s}{s + 1} \]

\[ x(t) = tu(t) \quad \Leftrightarrow \quad X(s) = \frac{1}{s^2} \]

\[ Y(s) = X(s)H(s) = \frac{s}{s + 1} \frac{1}{s^2} = \frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1} \]

Therefore,

\[ y(t) = (1 - e^{-t})u(t) \]