Lecture 12: Fourier Transform and Frequency Response of Systems

November 7, 2011
Today’s topics:

- Review
- Fourier Transforms of periodic signals
- Limiting transforms
Review

Properties of Fourier Transform

• **Shift**
  \[ x(t - t_0) \leftrightarrow X(j\omega)e^{-j\omega t_0} \]

• **Scaling**
  \[ x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \]

• **Derivative**
  \[ \frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega) \]

• **Modulation**
  \[ x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega - \omega_0)) \]

• **Duality**
  \[ X(t) \leftrightarrow 2\pi x(-\omega) \]
• Parseval
\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \]

• ★ The convolution theorem ★
\[ x(t) \ast y(t) \leftrightarrow X(j\omega)Y(j\omega) \]
Example.

Output is given by:

\[ y(t) = x(t) \ast h(t) \ast g(t) \]

This calculation is tedious.

In frequency domain, output is given by:

\[ Y(j\omega) = X(j\omega)H(j\omega)G(j\omega) \]

Analysis is often simpler
Example (Oppenheim & Willsky 4.14).

Consider a signal \( x(t) \) with Fourier transform \( X(j\omega) \). Suppose we are given the following facts.

- \( x(t) \) is real and nonnegative.
- \( \mathcal{F}^{-1}\{ (1 + j\omega)X(j\omega) \} = Ae^{-2t}u(t), \) where \( A \) is independent of \( t \).
- \( \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = 2\pi \)

Determine a closed-form expression for \( x(t) \).
Solution:
Start with fact #2:

\[
\mathcal{F}^{-1}\{(1 + j\omega)X(j\omega)\} = Ae^{-2t}u(t)
\]

Take the Fourier transform of both sides:

\[
\mathcal{F}\{\mathcal{F}^{-1}\{(1 + j\omega)X(j\omega)\}\} = \mathcal{F}\{Ae^{-2t}u(t)\}
\]

\[
(1 + j\omega)X(j\omega) = \frac{A}{2 + j\omega}
\]

\[
X(j\omega) = \frac{A}{(2 + j\omega)(1 + j\omega)}
\]

We would like to find an expression for \(x(t)\).
Currently \(X(j\omega)\) is not in the form whose inverse FT we know right away.
Use \textit{partial fractions} to convert $X(j\omega)$ into a more familiar form.

$$X(j\omega) = \frac{A}{(2 + j\omega)(1 + j\omega)} = \frac{a}{2 + j\omega} + \frac{b}{1 + j\omega}$$

with unknown constants $a$ and $b$.

$$X(j\omega) = \frac{A}{(2 + j\omega)(1 + j\omega)} = \frac{a(1 + j\omega) + b(2 + j\omega)}{(2 + j\omega)(1 + j\omega)}$$

Then

$$A = a(1 + j\omega) + b(2 + j\omega)$$

Since $A$ is a constant, it must be that $a + 2b = A$ and $a + b = 0$. Then $a = -A$ and $b = A$. Then we can write:

$$X(j\omega) = A\left(\frac{1}{1 + j\omega} + \frac{-1}{2 + j\omega}\right)$$
In time domain:

\[ x(t) = A(e^{-t} - e^{-2t})u(t) \]

To completely specify \( x(t) \), we need to find the value of \( A \).

Recall fact #3:

\[ \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \]

By Parseval’s relation:

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |x(t)|^2 dt \]

which means that

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = 1 \]

\[ A^2 \int_{0}^{\infty} |e^{-t} - e^{-2t}|^2 dt = A^2 \int_{0}^{\infty} (e^{-2t} - 2e^{-3t} + e^{-4t}) dt = 1 \]

\[ A^2 \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{A^2}{12} = 1 \]
\[ A = \sqrt{12} \]

Then the closed form expression for \( x(t) \) is:

\[ x(t) = \sqrt{12}(e^{-t} - e^{-2t})u(t) \]
Another version of convolution theorem

We’ve studied:

\[ x(t) * y(t) \leftrightarrow X(j\omega)Y(j\omega) \]

Convolution in time is equivalent to multiplication in frequency domain

Another version:

\[ \frac{1}{2\pi} X(j\omega) * Y(j\omega) \leftrightarrow x(t)y(t) \]

Convolution in frequency domain is equivalent to multiplication in time domain
Suppose $Z(j\omega) = X(j\omega) \ast Y(j\omega)$:

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\omega)e^{j\omega t} d\omega$$

$$Z(j\omega) = X(j\omega) \ast Y(j\omega) = \int_{-\infty}^{\infty} X(j\theta)Y(j(\omega - \theta))d\theta$$

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} X(j\theta)Y(j(\omega - \theta))d\theta \right)e^{j\omega t} d\omega$$

Rearrange the integrals:

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) \left( \int_{-\infty}^{\infty} Y(j(\omega - \theta))e^{j\omega t} d\omega \right) d\theta$$

Do "u-substitution". Let $u = \omega - \theta$

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) \left( \int_{-\infty}^{\infty} Y(ju)e^{j(u+\theta)t} du \right) d\theta$$
\[ z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)e^{j\theta t} d\theta \int_{-\infty}^{\infty} Y(ju)e^{jut} du \]

\[ \frac{1}{2\pi} z(t) = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)e^{j\theta t} d\theta \right) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(ju)e^{jut} du \right) \]

\[ \frac{1}{2\pi} z(t) = x(t)y(t) \]

Hence,

\[ \frac{1}{2\pi} X(j\omega) \ast Y(j\omega) \leftrightarrow x(t)y(t) \]

This is a generalization of modulation property!
Fourier Transform of Periodic Signals

We’ve seen that periodic signals can be represented using harmonically related exponentials (Fourier Series representation):

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

Frequency spectrum is discrete - only frequencies present in the signal are multiples of \( \omega_0 \).

Aperiodic signals can be represented using Fourier Transform

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, dt \]

Frequency spectrum is continuous - all frequencies could be present in the signal

EE102: Systems and Signals; Fall 2011, Jin Hyung Lee
Would like to find Fourier Transform representation of periodic signals.

Consider signal with \( X(j\omega) = 2\pi \delta(\omega - \omega_0) \).

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(2\pi \delta(\omega - \omega_0)\right) e^{j\omega t} \, dt
\]

\[
x(t) = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} \, dt
\]

\[
x(t) = e^{j\omega_0 t}
\]

Complex exponential at frequency \( \omega_0 \) is a delta function in frequency domain!
In general, if $X(j\omega) = 2\pi \sum_{k=\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(2\pi \sum_{k=\infty}^{\infty} a_k \delta(\omega - k\omega_0)\right) e^{j\omega t} dt$$

$$x(t) = \int_{-\infty}^{\infty} \sum_{k=\infty}^{\infty} a_k \delta(\omega - k\omega_0) e^{j\omega t} dt$$

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier transform of a periodic signal is a sequence of impulses in frequency domain. 
"Strength" of impulses is $2\pi$ larger than corresponding Fourier Series coefficients.
Example.
Fourier series coefficients of $x(t) = \cos(\omega_0 t)$ are:

$$a_k = \begin{cases} 
\frac{1}{2} & \text{if } k = \pm 1 \\
0 & \text{otherwise}
\end{cases}$$

Fourier transform of $x(t)$ is:

$$X(j\omega) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$
Fourier series coefficients of $x(t) = \sin(\omega_0 t)$ are:

$$a_k = \begin{cases} 
\frac{1}{2j} & \text{if } k = 1 \\
-\frac{1}{2j} & \text{if } k = -1 \\
0 & \text{otherwise}
\end{cases}$$

Fourier transform of $x(t)$ is:

$$X(j\omega) = \frac{\pi}{j} \left( \delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right)$$

Hint: $1/j = -j$
Example.
What is the Fourier transform of an impulse train?

\[ x(t) = \sum_k \delta(t - k \frac{2\pi}{\omega_0}) \]

What are the Fourier coefficients?
Solution: From lecture 7, we have determined that the Fourier Series coefficients of an impulse train is $a_k = \frac{1}{T}$ for all $k$. In other words,

$$x(t) = \sum_n \delta(t - nT) = \sum_k a_k e^{jk\omega_0 t} = \frac{1}{T} \sum_k e^{jk\frac{2\pi}{T} t}$$

(1)

Now, let $T = \frac{2\pi}{\omega_0}$, we have:

$$x(t) = \sum_k \delta(t - k\frac{2\pi}{\omega_0}) = \sum_k \delta(t - kT)$$

Applying the time shifting theorem, we obtain:

$$\hat{\delta}(t - kT) = e^{-j\omega kT} \hat{\delta}(t) = e^{-j\omega kT}$$
Therefore,

\[ \mathcal{F}[\sum_k \delta(t - kT)] = \sum_k e^{-j\omega kT} \]

\[ = \sum_k e^{-j\omega k \frac{2\pi}{\omega_0}} \]

\[ = \omega_0 \sum_k \delta(\omega - k\omega_0) \]
Limiting Transforms

Sometimes we want to find a transform for a signal for which the integral doesn’t converge, and there is no obvious indirect approach.

Another alternative is to represent the signal as a limit of a sequence of signals for which the Fourier transforms do exist,

\[ x_n(t) \xrightarrow{n \to \infty} x(t) \]

Then \( X(j\omega) = \lim_{n \to \infty} X_n(j\omega) \) is a reasonable definition for \( X(j\omega) \) if the limit makes sense.

**Example:** find the Fourier transform of the signum or sign signal

\[ x(t) = \text{sgn}(t) = \begin{cases} 
1 & t > 0 \\
0 & t = 0 \\
-1 & t < 0 
\end{cases} \]
We can approximate $x(t)$ by the signal

$$x_a(t) = e^{-at}u(t) - e^{at}u(-t)$$

as $a \to 0$. This looks like

As $a \to 0$, $x_a(t) \to \text{sgn}(t)$. 
The Fourier transform of $x_a(t)$ is

$$X_a(j\omega) = \mathcal{F}\{x_a(t)\}$$

$$= \mathcal{F}\{e^{-at}u(t) - e^{at}u(-t)\}$$

$$= \mathcal{F}\{e^{-at}u(t)\} - \mathcal{F}\{e^{at}u(-t)\}$$

$$= \frac{1}{a + j\omega} - \frac{1}{a - j\omega}$$

$$= \frac{-2j\omega}{a^2 + \omega^2}$$
If $\omega = 0$, then $X_a(j\omega) = 0$ for any $a \neq 0$. If $a > 0$, and $a \to 0$ then

$$
\lim_{a \to 0} X_a(j\omega) = \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{-2j\omega}{\omega^2} = \frac{2}{j\omega}
$$

This suggests we define the Fourier transform of $\text{sgn}(t)$ as

$$
\text{sgn}(t) \Leftrightarrow \begin{cases} 
\frac{2}{j\omega} & \omega \neq 0 \\
0 & \omega = 0 
\end{cases}
$$
With this, we can find the Fourier transform of the unit step,

\[ u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t) \]

as can be seen from the plots

\[ \begin{align*}
\text{sgn}(t) & \quad \text{for} \quad t < 0 \\
& \quad \text{for} \quad t \geq 0
\end{align*} \]

\[ \begin{align*}
u(t) & \quad \text{for} \quad t < 0 \\
& \quad \text{for} \quad t \geq 0
\end{align*} \]

The Fourier transform of the unit step is then

\[ \mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{1}{2} + \frac{1}{2} \text{sgn}(t)\right\} \]

\[ \mathcal{F}\{u(t)\} = \frac{1}{2} \left(2\pi \delta(\omega)\right) + \frac{1}{2} \left(\frac{2}{j\omega}\right) = \pi \delta(\omega) + \frac{1}{j\omega} \]
where the second term is replaced by zero at $\omega = 0$.

The transform pair is then

$$u(t) \Leftrightarrow \pi \delta(\omega) + \frac{1}{j\omega}$$

$$\mathbb{F}\{u(t)\} = \begin{cases} 
\frac{1}{j\omega} & \omega \neq 0 \\
\frac{\omega}{\pi} & \omega = 0 
\end{cases}.$$
Frequency Response of Systems Characterized by LCCDE

Linear constant coefficient differential equations have a form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

We will develop methods for solving these equations efficiently in frequency domain.

Recall that input/output relationship of an LTI system is described by $y(t) = x(t) * h(t)$ or in frequency domain by $Y(j\omega) = X(j\omega)H(j\omega)$.

We’d like to develop a similar expression for LTI systems described by LCCDE
Apply Fourier transform to both sides:

\[
\mathcal{F}\left\{ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} \right\} = \mathcal{F}\left\{ \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \right\}
\]

From linearity

\[
\sum_{k=0}^{N} a_k \mathcal{F}\left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{ \frac{d^k x(t)}{dt^k} \right\}
\]

Recall the derivative property of FT: \( \frac{d^n x(t)}{dt^n} \leftrightarrow (j\omega)^n X(j\omega) \). Then

\[
\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)
\]
Equivalently:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k}$$

The transfer function $H(j\omega)$ is a rational function, a polynomial in $(j\omega)$.

Key point:
If a system has an input/output relationship given by LCCDE, this method provides a simple way to characterize the system by $H(j\omega)$ and find impulse response $h(t)$.
Example.
A stable, causal LTI system is characterized by the differential equation (with $a > 0$):

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Using the derivative property of FT:

$$(j\omega)Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega}$$

This is easily recognized as

$$h(t) = e^{-at}u(t)$$

Process of solving the differential equation is reduced to an algebraic procedure
**Example.**
A causal, stable LTI system has the frequency response

\[ H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} \]

Find the corresponding LCCDE
Solution:

\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} \]

This is equivalent to:

\[ (6 - \omega^2 + 5j\omega)Y(j\omega) = (j\omega + 4)X(j\omega) \]

Using the derivative property, we obtain the following LCCDE

\[ \frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t) \]
We’ve seen before that an impulse response $h(t)$ completely characterizes an LTI system.

In many cases, knowing a single input/output pair (that is $x(t)$ and corresponding $y(t)$) is enough to fully characterize an LTI system, even if $x(t) \neq \delta(t)$ and $y(t) \neq h(t)$.

The following two examples illustrate the sufficient condition for doing so.
Example.
Suppose the input to the systems given below is $x(t) = \cos(t)$. What is the corresponding output?

a) $h_1(t) = -2\delta(t) + 5e^{-2t}u(t)$

b) $h_2(t) = 2te^{-t}u(t)$

Can $x(t) = \cos(t)$ be used as an input to fully characterize the system?
Solution: \( x(t) = \cos(t) = \frac{1}{2} [e^{jt} + e^{-jt}] \)

\[
H_1(j\omega) = -2 + \frac{5}{2 + j\omega}
\]

\[
\Rightarrow y_1(t) = \frac{1}{2} [H_1(j1)e^{jt} + H_1(-j1)e^{-jt}]
\]

\[
= -2 \cos(t) + \frac{1}{2} \left[ \frac{5\sqrt{5}}{3} e^{-j0.46} e^{jt} + \frac{5\sqrt{5}}{3} e^{j0.46} e^{-jt} \right]
\]

\[
= -2 \cos(t) + \frac{5\sqrt{5}}{3} \cos(t - 0.46)
\]
\[ H_2(j\omega) = \frac{2}{(1 + j\omega)^2} \]

\[ \Rightarrow y_2(t) = \frac{1}{2} \left[ H_2(j1)e^{jt} + H_2(-j1)e^{-jt} \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{j}e^{jt} - \frac{1}{j}e^{-jt} \right] \]
\[ = \sin(t) \]

\[ x(t) = \cos(t) \] cannot be used as an input to fully characterize the system because it only has one specific frequency, which is \( \omega_0 = 1 \).
**Example.**
Suppose you know the input to the system is $x(t) = e^{-t}u(t)$ and corresponding output $y(t) = e^{-5t}u(t)$.
What is $H(j\omega)$?

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\frac{1}{5+j\omega}}{\frac{1}{1+j\omega}}$$

$$H(j\omega) = \frac{1 + j\omega}{5 + j\omega}$$

For which inputs will this method fail?