1. (a) 
\[
\sum_{n=-\infty}^{\infty} |F_n|^2 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt \\
= \frac{1}{2} \int_{-1}^{1} t^4 dt = \frac{1}{5}
\]

(b) 
\[
\bar{\epsilon}_1^2 = \sum_{n=-\infty, n \neq -1, 0, 1}^{\infty} |F_n|^2 \\
= \sum_{n=-\infty}^{\infty} |F_n|^2 - (|F_{-1}|^2 + |F_0|^2 + |F_1|^2).
\]

From Hw5 we have \(F_0 = \frac{1}{3}\), and 
\[
F_n = \frac{2(-1)^n}{(n\pi)^2} \text{ for } n \neq 0.
\]

Therefore, 
\[
F_{-1} = F_1 = -\frac{2}{\pi^2} \\
\Rightarrow \bar{\epsilon}_1^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 - 2\left(\frac{2}{\pi^2}\right)^2 \\
= 6.67 \times 10^{-3}
\]

2. (a) 
\[
g(t) = f(t + \tau) = \sum_{n=-\infty}^{\infty} F_n e^{i\omega_0 (t + \tau)} \\
= \sum_{n=-\infty}^{\infty} F_n e^{i\omega_0 \tau} e^{i\omega_0 t} \\
\Rightarrow G_n = F_n e^{i\omega_0 \tau}
\]

(b) In class we found that the Fourier coefficients of the impulse train 
\[
f(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \text{ were } F_n = \frac{1}{T} \forall n.
\]
Taking \( T = 2 \), we see that \( g(t) \) is the shifted impulse train \( g(t) = f(t+1) \).

Therefore, applying part (a) with \( \omega_0 = \pi \) and \( \tau = 1 \) gives

\[
G_n = F_n e^{in\pi} = \frac{(-1)^n}{2}.
\]

Alternatively, we could integrate directly to get the \( G_n \), namely

\[
G_n = \frac{1}{2} \int_0^2 \delta(t-1)e^{-in\pi t}dt = \frac{1}{2} e^{-in\pi} = \frac{(-1)^n}{2}.
\]

\[
\implies g(t) = \sum_{n=-\infty}^{\infty} \left( -\frac{1}{2} \right)^n e^{in\pi t}
\]

(c) The simplest way is to use part (a), with \( \tau = \frac{T}{2} \). We have:

\[
G_n = F_n e^{in\omega_0 \frac{T}{2}} = F_n e^{in\pi} = F_n (-1)^n.
\]

But by hypothesis, \( g(t) = -f(t) \) so \( G_n = -F_n \), and we get

\[
(-1)^n F_n = -F_n.
\]

For odd values of \( n \), this says nothing new, but if \( n \) is even we get \( F_n = -F_n \) which means \( F_n = 0 \).

An alternative (longer) route is to show this by integration:

\[
F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-in\omega_0 t}dt
\]

\[
= \frac{1}{T} [\int_{-\frac{T}{2}}^{0} f(t)e^{-in\omega_0 t}dt + \int_{0}^{\frac{T}{2}} f(t)e^{-in\omega_0 t}dt]
\]

Let \( \tau = t + \frac{T}{2} \),

\[
F_n = \frac{1}{T} [\int_{0}^{\frac{T}{2}} f(\tau - \frac{T}{2})e^{-in\omega_0 (\tau - \frac{T}{2})}d\tau + \int_{\frac{T}{2}}^{T} f(t)e^{-in\omega_0 t}dt]
\]

\[
= \frac{1}{T} [\int_{0}^{\frac{T}{2}} -f(\tau)e^{-in\omega_0 \tau} e^{-in\pi} d\tau + \int_{\frac{T}{2}}^{T} f(t)e^{-in\omega_0 t}dt]
\]

\[
= \frac{1}{T} [(-1)^n \int_{0}^{\frac{T}{2}} f(\tau)e^{-in\omega_0 \tau} d\tau + \int_{\frac{T}{2}}^{T} f(t)e^{-in\omega_0 t}dt]
\]

\[
= 0 \quad \text{if } n \text{ is even.}
\]
d) Differentiating the Fourier expansion of $f(t)$, we get

$$\frac{df(t)}{dt} = \sum_{n=-\infty}^{\infty} F_n \frac{d}{dt} e^{in\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} (in\omega_0) F_n e^{in\omega_0 t}$$

therefore,

$$F_n^\prime = (in\omega_0) F_n$$

e) The plot shows the graphs of $f(t)$, $\frac{df(t)}{dt}$ and $\frac{d^2 f(t)}{dt^2}$. 
Note that
\[
\frac{d^2 f(t)}{dt^2} = 2 - 4g(t) \quad \text{(from part b)}
\]
\[
= 2 - 4 \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2} e^{in\pi t}
\]
\[
= \sum_{n=-\infty}^{\infty} F_n'' e^{in\pi t}
\]
Hence,
\[
F_n'' = \begin{cases} 
0 & \text{for } n = 0 \\
-2(-1)^n & \text{otherwise}.
\end{cases}
\]
Now,
\[
F''_n = (in\pi)F'_n = (in\pi)^2 F_n,
\]
so for \(n \neq 0\) we obtain
\[
F_n = \frac{-2(-1)^n}{-n^2\pi^2} = \frac{2(-1)^n}{n^2\pi^2},
\]
which agrees with the answer from Hw5.

\(F_0\) cannot be found from equation (1), since it is multiplied by zero. Another way to see this is that the DC value of \(f(t)\) can never be deduced from its derivatives, since constants go away when we differentiate.

3. (a)

\[
x(t) = \cos^3(t)
\]
\[
= \left(\frac{e^{it} + e^{-it}}{2}\right)^3
\]
\[
= \frac{1}{8} \left( e^{3it} + 3e^{it} + 3e^{-it} + e^{-3it} \right)
\]
\[
= \frac{1}{4} \left( \frac{e^{3it} + e^{-3it}}{2} + 3 \frac{e^{it} + e^{-it}}{2} \right)
\]
\[
= \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t)
\]
The above gives both the complex and the sine-cosine Fourier expansions.

(b) The RMS value of \(x(t)\),

\[
\|x\|^2 = \sqrt{\sum_{n=-\infty}^{\infty} |F_n|^2}
\]
\[
= \sqrt{|F_{-1}|^2 + |F_1|^2 + |F_{-3}|^2 + |F_3|^2}
\]
\[
= \sqrt{\left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \frac{1}{8^2} + \frac{1}{8^2}}
\]
\[
= \frac{\sqrt{5}}{4}
\]
(c) Since,

\[ x(t) = \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t) \]

The system output \( y(t) \) is

\[ y(t) = \frac{1}{4} |H(i3)| \cos(3t + \theta_1) + \frac{3}{4} |H(i1)| \cos(t + \theta_2) \]

where \( \theta_1 \) and \( \theta_2 \) are the phase of \( H(i3) \) and \( H(i1) \) respectively.

In order for \( y(t) \) to be sinusoidal, one of the two sets of conditions has to be satisfied.

\[
\left\{ \begin{array}{c}
H(i1) = 0 \\
H(i3) \neq 0 
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{c}
H(i1) \neq 0 \\
H(i3) = 0
\end{array} \right.
\tag{2}
\]

Now, \( H(i\omega) = 0 \) if and only if the numerator is zero, i.e.

\[ b - \omega^2 + a\omega i = 0. \]

Therefore, for the first alternative in (2) we must have \( a = 0 \) and \( b = 1 \).

For the second alternative, \( a = 0 \) and \( b = 9 \).