1. Consider the system $S$ described by the differential equation
\[
\frac{d^2}{dt^2} y(t) + 4 \frac{d}{dt} y(t) + 4 y(t) = \frac{d}{dt} x(t),
\]
with all-zero initial conditions.

(a) 4 points Write down the system function, $H(s)$.
Laplace: $Y(s)(s^2 + 4s + 4) = sX(s)$, which means $H(s) = \frac{s}{s^2 + 4s + 4} = \frac{s}{(s + 2)^2}$.

(b) 4 points Can the frequency response function, $H(i\omega)$, be obtained as $H(i\omega) = H(s)|_{s = i\omega}$? Explain.
Yes, because the poles of $H(s)$, $s = -2$, have negative real part. $H(i\omega) = \frac{i\omega}{(i\omega + 2)^2}$.

(c) 6 points Derive the impulse response function, $h(t)$.
$H(s) = \frac{A}{s + 2} + \frac{B}{(s + 2)^2}$; $B = (s + 2)^2 H(s)|_{s = -2} = -2$; $0 = H(0) = \frac{4}{2} - \frac{1}{2}$; hence $A = 1$. Therefore (use the tables!), $h(t) = U(t)(e^{-2t} - 2te^{-2t})$.

(d) 6 points Compute the output, $y(t)$, corresponding to the input $x(t) = \cos(t)$.
$y(t) = \frac{1}{2} e^{it} H(i) + \frac{1}{2} e^{-it} H(-i) = \frac{1}{2} e^{it} - \frac{i}{(i+2)^2} + \frac{1}{2} e^{-it} - \frac{i}{-i+2)^2}$, therefore $y_2(t) = \Re \left( e^{it} - \frac{i}{(i+2)^2} \right) = \Re \left( e^{it} \frac{4+3i}{25} \right)$, hence $y(t) = \frac{4}{25} \cos(t) - \frac{3}{25} \sin(t)$.

2. Consider the signal $x(t)$, periodic of period $T = 2$, defined as $x(t) = \sin \left( \frac{\pi}{2} t \right)$, $-1 < t < 1$.
(a) 3 points Consider its Fourier series coefficients, $X_n$. Do you expect these coefficients to be purely real or purely imaginary? Why? Answer without explicitly computing the coefficients.
Imaginary, because the function is real odd.
Write down the mathematical expression for $X(t)$. As a check, verify that the DC value, $X_0$, has the expected value (what is it?).

$$X_n = \frac{1}{2} \int_{-1}^{1} \sin \left( \frac{\pi}{2} t \right) e^{-i\pi n t} dt.$$ Use Euler’s formula for the $\sin(\cdot)$, and the equalities

$$e^{i\pi n/2} = \pm i, \quad e^{\pm i\pi} = (-1)^n.$$ Eventually you’ll get

$$X_n = \frac{i}{\pi} \left[ \sin \left( \frac{\pi}{2} (2n+1) \right) - \sin \left( \frac{\pi}{2} (2n-1) \right) \right] = \frac{i4n(-1)^n}{\pi(4n^2-1)}.$$ $X_0 = 0$, which is the average of the signal.

What is the mean power of $x(t)$, $\|x(t)\|_{\text{RMS}}^2$?

$$\|x(t)\|_{\text{RMS}}^2 = \frac{1}{2} \int_{-1}^{1} \sin^2 \left( \frac{\pi}{2} t \right) dt = \frac{1}{2}.$$ You can use the trigonometric equality

$$\sin^2(x) = \frac{1}{2} [1 - \cos(2x)].$$

Compute the mean square error, $\epsilon_1^2$, when approximating $x(t)$ up to its first harmonic. You are not required to carry out the calculations, just go as far as you can.

$$\epsilon_1^2 = \|x(t)\|_{\text{RMS}}^2 - |X_1|^2 - |X_0|^2 - |X_1|^2 = \frac{1}{2} - 2 \left( \frac{1}{3\pi} \right)^2 = \frac{1}{2} - \frac{32}{9\pi^2}.$$ Derive the expression for the signal, $x(t)$, input to a L, TI, C system with system function $H(s) = \frac{1}{s+1}$. What is the Fourier series expansion of the corresponding output, $y(t)$? No simplifications are required.

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{i\pi n t}, \quad \text{with} \quad Y_n = H(i\pi)X_n = \frac{1}{1+i\pi \pi(4n^2-1)}.$$
4. The following system

\[ x(t) \rightarrow \boxed{S_1} \rightarrow y(t) \rightarrow \boxed{S_2} \rightarrow z(t) \]

is composed of two L, TI, C stages.

If the input to the overall system is a unit step function, \( x(t) = U(t) \), then the output, \( z(t) \), is equal to \( z(t) = \left[ \cos(t) + 2 \sin(t) - e^{-\frac{1}{2}t} \right] U(t) \).

(a) **5 points** Compute the overall system function, \( H_{12}(s) \).

\[ H_{12}(s) = \frac{Z(s)}{X(s)} = \frac{s^2 + \frac{5}{2} s + \frac{5}{2}}{s^2 + \frac{5}{2} s + \frac{5}{2}}, \quad H_{12}(s) = \frac{5}{2} \frac{s^2}{(s^2 + 1)(s + \frac{1}{2})}, \quad \text{with region of convergence } \Re\{s\} > 0. \]

(b) **5 points** The system \( S_1 \) is described by the input output relationship

\[ y(t) = x(t) - \frac{1}{2} \int_{-\infty}^{t} e^{-\frac{1}{2}(t-\sigma)}x(\sigma) d\sigma. \]

What is the impulse response, \( h_1(t) \), of system \( S_1 \)?

\[ y(t) = \int_{-\infty}^{t} [\delta(t - \sigma) - \frac{1}{2} e^{-\frac{1}{2}(t-\sigma)}]x(\sigma) d\sigma, \quad \text{therefore } h_1(t) = \delta(t) - \frac{1}{2} e^{-\frac{1}{2}t} U(t). \]

(c) **5 points** Given the answers to parts (a) and (b), compute the system function, \( H_2(s) \), of system \( S_2 \).

\[ H_{12}(2) = H_1(s)H_2(s), \quad \text{therefore } H_2(s) = \frac{H_{12}(s)}{H_1(s)} \quad \text{where } H_1(s) = 1 - \frac{1}{2} \frac{1}{s + \frac{1}{2}} = \frac{s}{s + \frac{1}{2}}. \]

Hence, \( H_2(s) = \frac{5}{2} \frac{s}{s^2 + 1} \).

(d) **7 points** Compute the frequency response function, \( H_2(i\omega) \), of system \( S_2 \).

From the above, \( h_2(t) = \frac{5}{2} U(t) \cos(t) \), therefore \( H_2(i\omega) = \frac{5}{2} \frac{i\omega}{1 - \omega^2} + \frac{5\pi}{4} \delta(\omega + 1) + \frac{5\pi}{4} \delta(\omega - 1) \).
Consider now signal \( v(t) = e^{-|t|}, -\infty < t < \infty \). If \( v(t) \) is now the input to system \( S_2 \),

\[
v(t) \rightarrow S_2 \rightarrow w(t),
\]

what is the Fourier transform of the corresponding output, \( w(t) \)?

\[
V(i\omega) = \frac{2}{1+\omega^2}, \text{ therefore } W(i\omega) = V(i\omega)H_2(i\omega) = \frac{5i\omega}{1-\omega^2} + \frac{5\pi}{4}\delta(\omega+1) + \frac{5\pi}{4}\delta(\omega-1).
\]