Chapter 8. Wave Reflection and Transmission

8-1. Wave Reflection and Transmission at Normal Incidence.

At the boundary between two different types of media, the wave incident to this boundary will be partially reflected and partially transmitted, just like the wave encounters the discontinuity between two different transmission lines, as shown in Fig. 8-2.

To solve the reflection-transmission problem, we first write the solutions of the wave equation for three types of waves (incident, reflected, transmitted). Then apply the boundary conditions for tangential field components to match these solutions.

Figure 8-4: The two dielectric media separated by the x-y plane in (a) can be represented by the transmission-line analogue in (b).
Solutions of the three waves:

**Incident Wave:**
\[
\begin{align*}
\tilde{E}^i(z) &= \tilde{\hat{x}} E_0 e^{-j k z} \\
\tilde{H}^i(z) &= \tilde{\hat{z}} \times \frac{\tilde{E}^i(z)}{\eta_1} = \hat{\tau} \frac{E_0}{\eta_1} e^{-j k z}
\end{align*}
\]

**Reflected Wave:**
\[
\begin{align*}
\tilde{E}^r(z) &= \tilde{\hat{x}} E^r e^{j k z} \\
\tilde{H}^r(z) &= \tilde{\hat{z}} \times \frac{\tilde{E}^r(z)}{\eta_1} = -\hat{\tau} \frac{E^r}{\eta_1} e^{j k z}
\end{align*}
\]

**Transmitted Wave:**
\[
\begin{align*}
\tilde{E}^t(z) &= \tilde{\hat{x}} E^t e^{-j k z} \\
\tilde{H}^t(z) &= \tilde{\hat{z}} \times \frac{\tilde{E}^t(z)}{\eta_2} = \hat{\tau} \frac{E^t}{\eta_2} e^{-j k z}
\end{align*}
\]

Among these expressions, \( E^r, E^t \) are unknowns, \( E_0 \) is known incident field magnitude. One needs to solve \( E^r, E^t \) using boundary conditions. Also, we have,
\[
\begin{align*}
K_1 &= \omega \sqrt{\varepsilon_1} \\
K_2 &= \omega \sqrt{\varepsilon_2} \\
\eta_1 &= \sqrt{\frac{\varepsilon_1}{\varepsilon_1}} \\
\eta_2 &= \sqrt{\frac{\varepsilon_2}{\varepsilon_2}}
\end{align*}
\]

(\( \varepsilon_1, \varepsilon_2 \)) for medium 1, (\( \varepsilon_2, \varepsilon_2 \)) for medium 2.
The total electric field,

In Medium 1, sum of incident wave & reflected wave,

\[
\begin{align*}
\tilde{E}_1(z) &= \tilde{E}_i(z) + \tilde{E}_r(z) = \tilde{x}(E_0 e^{-jkz} + E_0 e^{jkz}) \\
\tilde{P}_1(z) &= \tilde{P}_i(z) + \tilde{P}_r(z) = \tilde{y}(E_0 e^{-jkz} - E_0 e^{jkz}) \frac{1}{\eta}
\end{align*}
\]

In Medium 2, transmitted wave only,

\[
\begin{align*}
\tilde{E}_2(z) &= \tilde{E}_t(z) = \tilde{x} E_0 e^{-jkrz} \\
\tilde{P}_2(z) &= \tilde{P}_t(z) = \tilde{y} \frac{E_0}{\eta_2} e^{-jkrz}
\end{align*}
\]

On the boundary, apply the tangential field continuity condition

\[
\begin{align*}
\tilde{E}_1(0) &= \tilde{E}_2(0) & \Rightarrow E_0^i + E_0^r &= E_0^t \\
\tilde{H}_1(0) &= \tilde{H}_2(0) & \Rightarrow \frac{E_0^i}{\eta_1} - \frac{E_0^r}{\eta_1} &= \frac{E_0^t}{\eta_2}
\end{align*}
\]

Simultaneous solutions of \( E_0^r \) and \( E_0^t \) in terms of \( E_0^i \) give,

\[
\begin{align*}
E_0^r &= \left( \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \right) E_0^i = PE_0^i \\
E_0^t &= \left( \frac{2\eta_2}{\eta_2 + \eta_1} \right) E_0^i = TE_0^i
\end{align*}
\]

Therefore, \( \Re = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \) (reflection coefficient)

\( \Te = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1} \) (transmission coefficient)

( \( \Re = 1 + \Re \) ), for nonmagnetic media (\( \mu = \mu_0 \)), one has

\[
\begin{align*}
\eta_1 &= \eta_0 / \Re e_1 \\
\eta_2 &= \eta_0 / \Re e_2 \Rightarrow \Re = \frac{\sqrt{\Re e_1} - \sqrt{\Re e_2}}{\sqrt{\Re e_1} + \sqrt{\Re e_2}}
\end{align*}
\]
8.1.2. Transmission-line Analogue

Based on transmission line concept, the reflection coefficient is,

\[ R = \frac{Z_{20} - Z_{01}}{Z_{20} + Z_{01}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \]

\((V, I, \beta, Z_0) \iff (E, H, k, \eta)\)

Transmission-line plane-wave

By analogy, the standing-wave ratio is given by

\[ S = \frac{|E_1|_{\text{max}}}{|E_1|_{\text{min}}} = \frac{1 + |P|}{1 - |P|} \]

Special cases:

1. **Match** \( \eta_1 = \eta_2, \) \( P = 0 \) and \( S = 1 \)
   
   Two media have the same \( \eta. \) (\( \varepsilon_r \) for nonmagnetic)

2. **Short circuit** \( \eta_2 = 0, \) \( P = -1 \) and \( S = \infty \)
   
   Media 2 is perfect electric conductor (PEC)

3. **Open circuit** \( \eta_2 = \infty, \) \( P = 1 \) and \( S = 0 \)
   
   Media 2 is perfect magnetic conductor (PMC)

which does not exist in nature.
Analog between plane wave and transmission line.

Plane-wave

\[ \vec{E}_1 \rightarrow \vec{E}_2 \]
\[ (k_1, \eta_1) \rightarrow (k_2, \eta_2) \]

\[ \vec{E}_1(z) = \vec{E}_0 e^{ik_2z} + \vec{p} e^{ik_0z} \]
\[ \vec{A}_1(z) = \frac{\vec{E}_0 \eta_1}{\eta_1} e^{-ik_2z} + \vec{p} e^{-ik_0z} \]
\[ \vec{E}_2(z) = \vec{A}_1(z) \cdot \vec{\tau} \cdot \vec{E}_0 e^{-ik_2z} \]
\[ \vec{A}_2(z) = \vec{A}_1(z) \cdot \vec{\tau} \cdot \vec{\eta}_2 e^{-ik_2z} \]
\[ p = \frac{(\eta_2 - \eta_1)}{(\eta_2 + \eta_1)} \]
\[ z = 1 + p \]
\[ k_1 = \omega \sqrt{\varepsilon_1}, \quad k_2 = \omega \sqrt{\varepsilon_2} \]
\[ \eta_1 = \sqrt{\varepsilon_1}, \quad \eta_2 = \sqrt{\varepsilon_2} \]

Transmission line

\[ \vec{I}(z) \rightarrow \vec{I}(z) \]
\[ \vec{V}(z) \rightarrow \vec{V}(z) \]
\[ (Z_{01}, \beta_1) \rightarrow (Z_{02}, \beta_2) \]

\[ \vec{V}_1(z) = V_0 e^{ik_2z} + \vec{P} e^{ik_0z} \]
\[ \vec{I}_1(z) = \frac{V_0}{Z_{01}} e^{-ik_2z} - \vec{P} e^{-ik_0z} \]
\[ \vec{V}_2(z) = \vec{Z} \vec{V}_0 e^{-ik_2z} \]
\[ \vec{I}_2(z) = \vec{Z} \vec{V}_0 e^{-ik_2z} \]

\[ p = \frac{(Z_{02} - Z_{01})}{(Z_{02} + Z_{01})} \]
\[ z = 1 + p \]
\[ \beta_1 = \omega \sqrt{\varepsilon_1}, \quad \beta_2 = \omega \sqrt{\varepsilon_2} \]
\[ Z_{01} = \frac{1}{\sqrt{\varepsilon_1}}, \quad Z_{02} = \frac{1}{\sqrt{\varepsilon_2}} \]
8-13. Power Flow in lossless Media

The average power density flow in medium 1 is,
\[ \overrightarrow{S_{av1}}(z) = \frac{1}{2} \Re \left[ \overrightarrow{E}(z) \times \overrightarrow{H}^*(z) \right] \]
\[ = \frac{1}{2} \Re \left[ \Re \left\{ E^2_0 e^{-j k_2 z} + P \Re \{ e^{j k_2 z} \} \right\} \times \nabla \cdot \frac{E^2_0}{\eta_1} e^{-j k_2 z} \right] \]
\[ = \frac{1}{2} \left[ E^2_0 \right] \left( 1 - |P|^2 \right) \]

Therefore, \( \overrightarrow{S_{av1}} = \overrightarrow{S_{av}} + \overrightarrow{S_{av}} \)
\[ \overrightarrow{S_{av}} = \frac{1}{2} \left[ E^2_0 \right] \left( \frac{1 - |P|^2}{\eta_1} \right) \]
\[ \overrightarrow{S_{av}} = - \frac{1}{2} |P|^2 \left[ E^2_0 \right] \]
\[ = - |P|^2 \overrightarrow{S_{av}} \]

The average power density of the transmitted wave in medium 2 is
\[ \overrightarrow{S_{av2}}(z) = \frac{1}{2} \Re \left[ \overrightarrow{E}(z) \times \overrightarrow{H}^*(z) \right] \]
\[ = \frac{1}{2} \Re \left[ \Re \left\{ E^2_0 e^{-j k_2 z} + \frac{1}{\eta_2} e^{j k_2 z} \right\} \times \nabla \cdot \frac{E^2_0}{\eta_2} e^{j k_2 z} \right] \]
\[ = \frac{1}{2} \left[ E^2_0 \right] \left( \frac{1 - |P|^2}{\eta_2} \right) \]
\[ \frac{E^2_0}{\eta_2} = \frac{1 - |P|^2}{\eta_1} \quad \text{(lossless media)} \]
\[ \Rightarrow \overrightarrow{S_{av1}} = \overrightarrow{S_{av2}} \quad \text{(energy conservation)} \]
Similarly to lossless cases, it is analogous to a transmission line problem, as shown in Fig. 8-7. We can obtain the results from extension of lossless cases.

\[
\begin{align*}
\text{lossless} & \quad \rightarrow \quad \text{lossy} \\
\varepsilon^{-3kz} & \quad \rightarrow \quad \varepsilon^{-k_1z} \\
\varepsilon^{-3kz} & \quad \rightarrow \quad \varepsilon^{-k_2z} \\
\eta_1 & \quad \rightarrow \quad \eta_1'c \\
\eta_2 & \quad \rightarrow \quad \eta_2'c
\end{align*}
\]

Therefore, the fields are

Medium 1:
\[
\begin{align*}
\vec{E}_1(z) &= \vec{E}_0 (e^{-k_1z} + Re^{k_1z}) \\
\vec{H}_1(z) &= \frac{\mu}{\eta_1} \vec{E}_0 (e^{-k_1z} - Re^{k_1z})
\end{align*}
\]

Medium 2:
\[
\begin{align*}
\vec{E}_2(z) &= \vec{E}_0 e^{-k_2z} \\
\vec{H}_2(z) &= \frac{\mu}{\eta_2} \vec{E}_0 e^{-k_2z}
\end{align*}
\]

\[\begin{align*}
\gamma_1 &= \alpha_1 + j\beta_1, \quad \gamma_2 &= \alpha_2 + j\beta_2 \quad \text{so finally,} \\
\rho &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad \tau = 1 + \rho = \frac{2\eta_2}{\eta_2 + \eta_1}
\end{align*}\]