Chapter 7

Section 7-2: Propagation in Lossless Media

Problem 7.1 The magnetic field of a wave propagating through a certain nonmagnetic material is given by

\[ \mathbf{H} = 2.50 \cos(10^9 t - 5y) \]  

\( \text{mA/m} \).

Find (a) the direction of wave propagation, (b) the phase velocity, (c) the wavelength in the material, (d) the relative permittivity of the material, and (e) the electric field phasor.

Solution:

(a) Positive y-direction.

(b) \( \omega = 10^9 \text{ rad/s}, \ k = 5 \text{ rad/m} \).

\[ u_p = \frac{\omega}{k} = \frac{10^9}{5} = 2 \times 10^8 \text{ m/s}. \]

(c) \( \lambda = \frac{2\pi}{k} = \frac{2\pi}{5} = 1.26 \text{ m}. \)

(d) \( \varepsilon_r = \left( \frac{c}{u_p} \right)^2 = \left( \frac{3 \times 10^8}{2 \times 10^8} \right)^2 = 2.25. \)

(e) From Eq. (7.39b),

\[ \mathbf{\vec{E}} = -\eta \mathbf{\hat{k}} \times \mathbf{\vec{H}}, \]

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{120\pi}{\sqrt{1.5 \varepsilon_r}} = 251.33 \]  

\( \text{(\Omega)}, \)

\[ \mathbf{\hat{k}} = \mathbf{\hat{y}}, \quad \text{and} \quad \mathbf{\vec{H}} = 250e^{-j5y} \times 10^{-3} \]  

\( \text{(A/m)}. \)

Hence,

\[ \mathbf{\vec{E}} = -251.33 \mathbf{\hat{y}} \times 250e^{-j5y} \times 10^{-3} = -\mathbf{\hat{x}} 12.57e^{-j5y} \]  

\( \text{(V/m)}, \)

and

\[ \mathbf{E}(y,t) = \Re(\mathbf{\vec{E}}e^{j\omega t}) = -\mathbf{\hat{x}} 12.57 \cos(10^9 t - 5y) \]  

\( \text{(V/m)}. \)

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Problem 7.2 Write general expressions for the electric and magnetic fields of a 1-GHz sinusoidal plane wave traveling in the +y-direction in a lossless nonmagnetic medium with relative permittivity \( \varepsilon_r = 9 \). The electric field is polarized along the x-direction, its peak value is 3 V/m and its intensity is 2 V/m at \( t = 0 \) and \( y = 2 \text{ cm} \).
Solution: For $f = 1$ GHz, $\mu_r = 1$, and $\varepsilon_r = 9,$
\[
\omega = 2\pi f = 2\pi \times 10^9 \text{ rad/s},
\]
\[
k = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_0} \sqrt{\varepsilon_r} = \frac{2\pi f}{c} \sqrt{\varepsilon_r} = \frac{2\pi \times 10^9}{3 \times 10^8} \sqrt{9} = 20\pi \text{ rad/m},
\]
\[
E(y,t) = \hat{x} 3 \cos(2\pi \times 10^9 t - 20\pi y + \phi_0) \quad (\text{V/m}).
\]
At $t = 0$ and $y = 2$ cm, $E = 2$ V/m:
\[
2 = 3 \cos(-20\pi \times 2 \times 10^{-2} + \phi_0) = 3 \cos(-0.4\pi + \phi_0).
\]
Hence,
\[
\phi_0 - 0.4\pi = \cos^{-1}\left(\frac{2}{3}\right) = 0.84 \text{ rad},
\]
which gives
\[
\phi_0 = 2.1 \text{ rad} = 120.19^\circ
\]
and
\[
E(y,t) = \hat{x} 3 \cos(2\pi \times 10^9 t - 20\pi y + 120.19^\circ) \quad (\text{V/m}).
\]

Problem 7.3 The electric field phasor of a uniform plane wave is given by $\mathbf{E} = \hat{y} 10e^{j0.2z}$ (V/m). If the phase velocity of the wave is $1.5 \times 10^8$ m/s and the relative permeability of the medium is $\mu_r = 2.4$, find (a) the wavelength, (b) the frequency $f$ of the wave, (c) the relative permittivity of the medium, and (d) the magnetic field $\mathbf{H}(z,t)$.

Solution:
(a) From $\mathbf{E} = \hat{y} 10e^{j0.2z}$ (V/m), we deduce that $k = 0.2$ rad/m. Hence,
\[
\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2} = 10\pi = 31.42 \text{ m}.
\]
(b)
\[
f = \frac{u_p}{\lambda} = \frac{1.5 \times 10^8}{31.42} = 4.77 \times 10^6 \text{ Hz} = 4.77 \text{ MHz}.
\]
(c) From
\[
u_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}, \quad \varepsilon_r = \frac{1}{\mu_r} \left(\frac{c}{u_p}\right)^2 = \frac{1}{2.4} \left(\frac{3}{1.5}\right)^2 = 1.67.
\]
(d) \[
\eta = \sqrt{\frac{\mu}{\varepsilon}} \approx 120\pi \sqrt{\frac{\mu_{r}}{\varepsilon_{r}}} = 120\pi \sqrt{\frac{2.4}{1.67}} = 451.94 \text{ (}\Omega\text{)},
\]
\[
\tilde{H} = \frac{1}{\eta} (\tilde{\mathbf{z}} \times \tilde{E}) = \frac{1}{\eta} (-\hat{\mathbf{z}}) \times \hat{\mathbf{y}} 10e^{j0.2\tau} = \hat{\mathbf{x}} 22.13e^{j0.2\tau} \text{ (mA/m)},
\]
\[
H(z, t) = \hat{\mathbf{x}} 22.13 \cos(\omega t + 0.2\tau) \text{ (mA/m)},
\]
with \( \omega = 2\pi f = 9.54\pi \times 10^6 \text{ rad/s}. \)

**Problem 7.4** The electric field of a plane wave propagating in a nonmagnetic material is given by
\[
E = [\hat{\mathbf{y}} 3 \sin(2\pi \times 10^7 t - 0.4\pi x) + \hat{\mathbf{x}} 4 \cos(2\pi \times 10^7 t - 0.4\pi x)] \text{ (V/m)}.\]

Determine (a) the wavelength, (b) \( \varepsilon_{r} \), and (c) \( H \).

**Solution:**
(a) Since \( k = 0.4\pi \),
\[
\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.4\pi} = 5 \text{ m}.
\]
(b) \[
\nu_p = \frac{\omega}{k} = \frac{2\pi \times 10^7}{0.4\pi} = 5 \times 10^7 \text{ m/s}.
\]
But
\[
\nu_p = \frac{c}{\sqrt{\varepsilon_{r}}},
\]
Hence,
\[
\varepsilon_{r} = \left( \frac{c}{\nu_p} \right)^2 = \left( \frac{3 \times 10^8}{5 \times 10^7} \right)^2 = 36.
\]
(c) \[
H = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\eta} \hat{\mathbf{k}} \times \left[ \hat{\mathbf{y}} 3 \sin(2\pi \times 10^7 t - 0.4\pi x) + \hat{\mathbf{x}} 4 \cos(2\pi \times 10^7 t - 0.4\pi x) \right] \]
\[
= \frac{3}{\eta} \sin(2\pi \times 10^7 t - 0.4\pi x) - \frac{4}{\eta} \cos(2\pi \times 10^7 t - 0.4\pi x) \text{ (A/m)},
\]
Problem 7.9. The electric field of a uniform plane wave propagating in free space is given by $\mathbf{E} = (x + jy)20e^{-j\pi/6} \text{ (V/m)}$. Specify the modulus and direction of the electric field intensity at the $z = 0$ plane at $t = 0, 5$ and $10 \text{ ns}$. 

Figure P7.8: Plots of the locus of $\mathbf{E}(0,t)$. 
Solution:

\[ E(z, t) = \Re\{E e^{j\omega t}\} \]
\[ = \Re\{(\hat{x} + j\hat{y})20e^{-j\pi z/6}e^{j\omega t}\} \]
\[ = \Re\{(\hat{x} + j\hat{y})e^{j\pi/2}20e^{-j\pi z/6}e^{j\omega t}\} \]
\[ = \hat{x}20\cos(\omega t - \pi z/6) + \hat{y}20\cos(\omega t - \pi z/6 + \pi/2) \]
\[ = \hat{x}20\cos(\omega t - \pi z/6) - \hat{y}20\sin(\omega t - \pi z/6) \] (V/m),
\[ |E| = \left[ E_x^2 + E_y^2 \right]^{1/2} = 20 \] (V/m),
\[ \psi = \tan^{-1} \left( \frac{E_x}{E_y} \right) = -(\omega t - \pi z/6). \]

From

\[ f = \frac{c}{\lambda} = \frac{k c}{2\pi} = \frac{\pi/6 \times 3 \times 10^8}{2\pi} = 2.5 \times 10^7 \text{ Hz}, \]
\[ \omega = 2\pi f = 5\pi \times 10^7 \text{ rad/s}. \]

At \( z = 0 \),
\[ \psi = -\omega t = -5\pi \times 10^7 t = \begin{cases} 0 & \text{at } t = 0, \\ -0.25\pi = -45^\circ & \text{at } t = 5 \text{ ns}, \\ -0.5\pi = -90^\circ & \text{at } t = 10 \text{ ns}. \end{cases} \]

Therefore, the wave is LHC polarized.

**Problem 7.10** A linearly polarized plane wave of the form \( \vec{E} = \hat{x}a_x e^{-j kz} \) can be expressed as the sum of an RHC polarized wave with magnitude \( a_R \) and an LHC polarized wave with magnitude \( a_L \). Prove this statement by finding expressions for \( a_R \) and \( a_L \) in terms of \( a_x \).

Solution:

\[ \vec{E} = \hat{x}a_x e^{-j kz}, \]

RHC wave: \( \vec{E}_R = a_R(\hat{x} + j\hat{y})e^{-j\pi/2}e^{-j kz} = a_R(\hat{x} - j\hat{y})e^{-j kz}, \)

LHC wave: \( \vec{E}_L = a_L(\hat{x} + j\hat{y})e^{-j\pi/2}e^{-j kz} = a_L(\hat{x} + j\hat{y})e^{-j kz}, \)

\[ \vec{E} = \vec{E}_R + \vec{E}_L, \]
\[ \hat{x}a_x = a_R(\hat{x} - j\hat{y}) + a_L(\hat{x} + j\hat{y}). \]

By equating real and imaginary parts, \( a_x = a_R + a_L, \ 0 = -a_R + a_L, \) or \( a_L = a_x/2, \)
\[ a_R = a_x/2. \]
Problem 7.11  The electric field of an elliptically polarized plane wave is given by

\[ E(z, t) = [-\hat{x} 10 \sin(\omega t - kz - 60^\circ) + \hat{y} 20 \cos(\omega t - kz)] \quad (\text{V/m}). \]

Determine (a) the polarization angles \((\gamma, \chi)\) and (b) the direction of rotation.

Solution:

(a)

\[ E(z, t) = [-\hat{x} 10 \sin(\omega t - kz - 60^\circ) + \hat{y} 20 \cos(\omega t - kz)] \]
\[ = [\hat{x} 10 \cos(\omega t - kz + 30^\circ) + \hat{y} 20 \cos(\omega t - kz)] \quad (\text{V/m}). \]

Phasor form:

\[ \vec{E} = (\hat{x} 10e^{j30^\circ} + \hat{y} 20)e^{-jkz}. \]

Since \(\delta\) is defined as the phase of \(E_y\) relative to that of \(E_x\),

\[ \delta = -30^\circ, \]
\[ \psi_0 = \tan^{-1}\left(\frac{20}{10}\right) = 63.44^\circ, \]

\[ \tan 2\gamma = (\tan 2\psi_0) \cos \delta = -1.15 \quad \text{or} \quad \gamma = 65.5^\circ, \]

\[ \sin 2\chi = (\sin 2\psi_0) \sin \delta = -0.40 \quad \text{or} \quad \chi = -11.79^\circ. \]

(b) Since \(\chi < 0\), the wave is right-hand elliptically polarized.

Problem 7.12  Compare the polarization states of each of the following pairs of plane waves:

(a) wave 1: \( E_1 = \hat{x} 2 \cos(\omega t - kz) + \hat{y} 2 \sin(\omega t - kz) \),
wave 2: \( E_2 = \hat{x} 2 \cos(\omega t + kz) + \hat{y} 2 \sin(\omega t + kz) \).

(b) wave 1: \( E_1 = \hat{x} 2 \cos(\omega t - kz) - \hat{y} 2 \sin(\omega t - kz) \),
wave 2: \( E_2 = \hat{x} 2 \cos(\omega t + kz) - \hat{y} 2 \sin(\omega t + kz) \).

Solution:

(a)

\[ E_1 = \hat{x} 2 \cos(\omega t - kz) + \hat{y} 2 \sin(\omega t - kz) \]
\[ = \hat{x} 2 \cos(\omega t - kz) + \hat{y} 2 \cos(\omega t - k\pi/2), \]
\[ \vec{E}_1 = \hat{x} 2e^{-jkz} + \hat{y} 2e^{-jk\pi/2}. \]