ELECTROSTATICS

Total charge $Q = \sum q_i$ (discrete charges)

$$= \int \rho_v dv \quad \rho_v: \text{volume charge density}$$

Coulomb’s Law

$$\vec{E} = \hat{R} \frac{q}{4\pi \varepsilon R^2} \quad (\text{V/m})$$

$$\vec{D} = \varepsilon \vec{E} \quad (\text{C/m}^2) \quad (\text{Flux density}), \quad \varepsilon = \varepsilon_r \varepsilon_0 \quad (\text{F/m})$$

$\varepsilon_r: \text{Dielectric constant (or Relative permittivity)}$

$\varepsilon_0 = 8.85 \times 10^{-12}$

For a collection of discrete charges

$$\vec{E} = \sum_i \vec{E}_i \quad \vec{E}_i = \hat{R}_i \frac{q_i}{4\pi \varepsilon R_i^2} \quad (\text{Vector sum})$$

For charge distribution

$$\vec{E} = \int \vec{dE} = \frac{1}{4\pi \varepsilon} \int \hat{R} \frac{\rho_v}{R^2} dv$$
Gauss's Law

\[ \oint \mathbf{D} \cdot d\mathbf{s} = Q \]

Total flux coming out of a closed surface is equal to the total charge contained within.

Divergence Theorem gives \[ \oint \mathbf{D} \cdot d\mathbf{s} = \int \nabla \cdot \mathbf{D} \, dv \]

Charge is given by \[ Q = \int \rho_v \, dv \]

\[ \nabla \cdot \mathbf{D} = \rho_v \quad \text{(Gauss's Law in differential form)} \]

Usage of Gauss's Law: Useful in finding the electric field from charge distribution for special cases with some symmetry.

Coulomb's Law can be applied to any cases, but it may require complicated integration process.

The end results by two ways should be the same.
Electric Scalar Potential

\[ V_{21} = V_2 - V_1 = - \int_{P_1}^{P_2} \bar{E} \cdot d\bar{l} \]

Energy accumulation when moving a charge from \( P_1 \) to \( P_2 \).

\[ V = - \int_{\infty}^{P} \bar{E} \cdot d\bar{l} \quad \oint_{C} \bar{E} \cdot d\bar{l} = 0 \quad \text{(Electrostatic)} \]

Stokes Theorem gives

\[ \int_S (\nabla \times \bar{E}) \cdot ds = \oint_{C} \bar{E} \cdot d\bar{l} = 0 \implies \nabla \times \bar{E} = 0 \]

Potential due to point charges

\[ \bar{E} = \hat{R} \frac{q}{4\pi\varepsilon R^2} \]

\[ V = - \int_{\infty}^{R} \bar{E} \cdot d\bar{l} = - \int_{\infty}^{R} \hat{R} \frac{q}{4\pi\varepsilon R^2} \cdot \hat{R}dR = \frac{q}{4\pi\varepsilon R} \]

Distributed charges

\[ V(R) = \frac{1}{4\pi\varepsilon} \int_{\frac{1}{R}} \rho_v dv \quad \text{(Scalar integration)} \]
Electric Field and Electrostatic Potential

\[ V = \int_{\infty}^{P} dV = - \int_{\infty}^{P} \vec{E} \cdot d\vec{l} \]  
Hence, \( dV = - \vec{E} \cdot d\vec{l} \)

By definition, \( dV = \nabla V \cdot d\vec{l} \)

\[ \vec{E} = -\nabla V = \text{grad } V \]

Electric field is given by the change of \( V \) per length.

Poisson’s Equation

\[ \nabla \cdot \vec{D} = \rho_v \] (Gauss’s Law) and \( \vec{D} = \varepsilon \vec{E} \Rightarrow \)

\[ \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} \]  
Combine this with \( \vec{E} = -\nabla V \)

\[ \nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon} \]

Note that \( \nabla \cdot \nabla V = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \)

\[ \nabla^2 V = -\frac{\rho_v}{\varepsilon} \] (Poisson’s Equation)

\[ \nabla^2 V = 0 \] (Laplace Equation) if source free
Conductors

Ohm's Law in point form $\vec{J} = \sigma \vec{E}$, $\sigma$: conductivity

- Perfect dielectric $\vec{J} = 0$ ($\sigma = 0$)
- Perfect conductor $\vec{E} = 0$ ($\sigma = \infty$)

A perfect conductor is equi-potential. No potential difference throughout. Most good conductors are almost perfect.

Resistance

$\int \vec{E} \cdot d\vec{l} \quad I = \int \vec{J} \cdot ds$

$R = \frac{V}{I} = \frac{-\int \vec{E} \cdot d\vec{l}}{\int \vec{J} \cdot ds} = \frac{-\int \vec{E} \cdot d\vec{l}}{\int \sigma \vec{E} \cdot ds}$

For even current density and electric field,

$R = \frac{E_x l}{\sigma E_x S} = \frac{l}{\sigma S}$
Example of Coaxial Cable (Leakage Conductance)

$I$: Total current flowing from the inner conductor to the outer conductor through the insulator (leakage)

Leakage conductance $G = I/V_{ab}$

$$\bar{J} = \hat{r} \frac{I}{A} \quad \text{and} \quad A = 2\pi rl \rightarrow \bar{J} = \hat{r} \frac{I}{2\pi rl} = \sigma \bar{E}$$

$$\bar{E} = \hat{r} \frac{I}{2\pi \sigma rl}$$

Note $V_{ab} = -\int_{b}^{a} \bar{E} \cdot d\vec{l} = -\int_{b}^{a} \frac{I}{2\pi \sigma l} \hat{r} \cdot \hat{r} \, dr$

$$= \frac{I}{2\pi \sigma l} \ln \left( \frac{b}{a} \right)$$

Conductance per unit length $G'$ is

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab} l} = \frac{2\pi \sigma}{\ln(b/a)} \quad (\text{S/m})$$