2.1 Transmission Lines

Any structures or media that serve to transfer electrical energy or information between two points.

Examples: 
- Jump start cable (DC energy)
- Telephone wires (Voice signal)
- Coaxial cable (TV signal)
- Optical fiber (Optical signal)

Thevenin-equivalent generator circuit

Generator circuit  Load circuit

Important variables:
- $v_g(t)$: Generator voltage
- $R_g$: Generator resistance
- $R_L$: Load resistance
2.1.1 Role of Wavelength

If along the transmission line, the wave travels as
\[ \cos(\omega t - \beta z) = \cos[\omega t - 2\pi z/\lambda] \]
At \( t = 0 \), compare \( v_{AA} \) at \( z = 0 \) and \( v_{BB} \) at \( z = z \)

For \( z/\lambda < 0.01 \), \( v_{AA} \approx v_{BB} \).

Transmission line effect negligible
For \( z/\lambda > 0.01 \), wave phenomena cannot be neglected.

Transmission lines are characterized by the distributed parameters (capacitance, inductance).
They may cause

- Reflection \( \rightarrow \) Impedance mismatch
- Power loss \( \rightarrow \) Lossy medium, radiation
- Dispersion \( \rightarrow \) Frequency-dependent traveling speed of energy components

\[ \rightarrow \) No dispersion
\[ \rightarrow \) Short line, light dispersion
\[ \rightarrow \) Long line, heavy dispersion
2.1.2 Propagation Mode

(a) Transverse electromagnetic (TEM) transmission lines

- E and H fields are entirely transverse to propagation
- Field in the transverse direction similar to static
- Support DC power transmission
- No dispersion or little dispersion
- Typically two conductor system

(b) Higher order transmission lines

- At least one significant field component in the direction of propagation
- Does not support DC
- Strong dispersion
- Hollow waveguide (single conductor), optical fiber
2.2 Lumped-Element Model

Objective: To represent the distributed effect (capacitance, inductance, resistance, conductance) along the transmission line by a two-wire model.

This is a model for a two-wire transmission line where $\Delta z \rightarrow 0$.

$R'$: Combined resistance of both conductors per unit length in $\Omega/m$

$L'$: Combined inductance of both conductors per unit length in $H/m$

$G'$: Conductance of the insulation medium per unit length in $S/m$

$C'$: Capacitance between the two conductors per unit length in $F/m$
Examples of Transmission Line Parameters

<table>
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<th>Formula</th>
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<tr>
<td>Coaxial</td>
<td>$R' = \frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right)$</td>
</tr>
<tr>
<td>Two wire</td>
<td>$\frac{R_s}{\pi a}$</td>
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<tr>
<td>Parallel Plate</td>
<td>$\frac{2R_s}{w}$</td>
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<tr>
<td>Unit</td>
<td>$\Omega/m$</td>
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$L' = \frac{\mu}{2\pi} \ln\left( \frac{b}{a} \right)$  
$G' = \frac{2\pi\sigma}{\ln\left( \frac{b}{a} \right)}$  
$C' = \frac{2\pi\varepsilon}{\ln\left( \frac{b}{a} \right)}$

where $R_s = \sqrt{\pi f \mu_c / \sigma_c}$ is the surface resistance.

Interesting Relationships

$L' C' = \mu \varepsilon$  
$G'/C' = \sigma / \varepsilon$

If the medium is air

$\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$  
$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$  
$\sigma = 0$ so that $G' = 0$
2.3 Transmission Line Equations

Let us look at the “unit cell” of the transmission line.

Application of Kirchhoff’s voltage law accounts for the voltage drop across the series resistance $R'\Delta z$ and the inductance $L'\Delta z$:

$$v(z, t) = R'\Delta z \, i(z, t) + L'\Delta z \, \frac{\partial i(z, t)}{\partial t} + v(z+\Delta z, t)$$

From the above,

$$- \left[ \frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} \right] = R'i(z, t) + L'\frac{\partial i(z, t)}{\partial t}$$

When $\Delta z \to 0$

$$- \frac{\partial v(z, t)}{\partial z} = R'i(z, t) + L'\frac{\partial i(z, t)}{\partial t}$$
Application of Kirchhoff's current law at node $N+1$:

$$i(z, t) = -G' \Delta z \nu(z+\Delta z, t) - C' \Delta z \frac{\partial \nu(z, t)}{\partial t}$$

$$= i(z+\Delta z, t)$$

In the limit of $\Delta z \to 0$, we obtain

$$-\frac{\partial i(z, t)}{\partial z} = G' \nu(z, t) + C' \frac{\partial \nu(z, t)}{\partial t}$$

Hence, transmission line equations or telegrapher's equations are

$$-\frac{\partial \nu(z, t)}{\partial z} = R' i(z, t) + L' \frac{\partial i(z, t)}{\partial t}$$

$$-\frac{\partial i(z, t)}{\partial z} = G' \nu(z, t) + C' \frac{\partial \nu(z, t)}{\partial t}$$

For the time-harmonic case

$$\nu(z, t) = \text{Re}[\tilde{\nu}(z)e^{j\omega t}]$$ and $$i(z, t) = \text{Re}[\tilde{I}(z)e^{j\omega t}]$$

The phasor form is

$$-\frac{d\tilde{\nu}(z)}{dz} = (R' + j\omega L') \tilde{I}(z)$$ \hspace{1cm} (T1)

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C') \tilde{\nu}(z)$$ \hspace{1cm} (T2)
2.4 Wave Propagation on a Transmission Line

Take derivative of both sides of (T1) by $z$

$$-\frac{d^2 \tilde{V}(z)}{dz^2} = (R' + j\omega L') \frac{d\tilde{I}(z)}{dz}$$

Substitute (T2) into the above.

$$\frac{d^2 \tilde{V}(z)}{dz^2} - (R' + j\omega L')(G' + j\omega C')\tilde{V}(z) = 0$$

or

$$\frac{d^2 \tilde{V}(z)}{dz^2} - \gamma^2 \tilde{V}(z) = 0 \quad \text{(1-D wave equation)}$$

where

$$\gamma \equiv \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

Similarly,

$$\frac{d^2 \tilde{I}(z)}{dz^2} - \gamma^2 \tilde{I}(z) = 0$$

$\gamma = \alpha + j\beta$ : Complex propagation constant

$\alpha = \text{Re}(\gamma)$ : Attenuation constant

$\beta = \text{Im}(\gamma)$ : Phase constant
Solutions to the wave equations
\[ \tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \]
\[ \tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \]
Superscript + for the forward traveling wave and – for the backward traveling wave.

\( V_0^+, V_0^-, I_0^+, I_0^- \) are determined by the boundary conditions (conditions from the generator and load)

\( \gamma \) is characteristic to the transmission line.

Substitution of the above solutions to (T1) results in
\[ \tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \]
\[ = \frac{\gamma}{R' + j\omega L'} [V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}] \]
Hence,
\[ \frac{V_0^+}{I_0^+} = Z_0 = \frac{-V_0^-}{I_0^-} \]

where \( Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \frac{\gamma}{G' + j\omega C'} \)

\( Z_0 : \) Characteristic impedance

Definition: Ratio of the voltage amplitude to the current amplitude for each traveling wave term individually.
Using $Z_0$,

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

Let us derive the time domain form.

Let

$$V_0^+ = \| V_0^+ \| e^{j\phi}, \quad V_0^- = \| V_0^- \| e^{j\phi}$$

From the phasor definition,

$$v(z, t) = \text{Re}[\tilde{V}(z)e^{j\omega t}]$$

$$= \text{Re}[(V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z})e^{j\omega t}]$$

$$= \text{Re}[\| V_0^+ \| e^{j\phi} e^{j\omega t} e^{-(\alpha+j\beta)z} + \| V_0^- \| e^{j\phi} e^{j\omega t} e^{(\alpha+j\beta)z}]$$

$$= \| V_0^+ \| e^{-\alpha z} \cos(\omega t - \beta z + \phi_+) + \| V_0^- \| e^{\alpha z} \cos(\omega t + \beta z + \phi_-)$$

The phase velocity is

$$u_p = f\lambda = \frac{\omega}{\beta}$$